AN INTRODUCTION

TO

PHYSICAL MEASUREMENTS

AN INTRODUCTION

TO

PHYSICAL MEASUREMENTS

WITH APPENDICES ON

ABSOLUTE ELECTRICAL MEASUREMENT, ETC.

BY DR. F. KOHLRAUSCH

PROFESSOR-IN-ORDINARY AT THE UNIVERSITY OF STRASSBURG

THIRD EDITION

Translated from the Seventh German Edition

BY THOMAS HUTCHINSON WALLER, B.A., B.Sc.

AND

HENRY RICHARDSON PROCTER, F.I.C., F.C.S.

NEW YORK

D. APPLETON & CO.

1, 3 AND 5 BOND STREET

1894

PREFACE

THE Author, in the preface to the second German edition, gives a sketch of the purposes which he hopes that the present book will serve. He says, a truth which all experience confirms, that the mere verbal teaching of physical laws is seldom of much use, tending frequently merely to confuse the student; while the simple performance of an experiment gives him confidence in himself and in the laws he is investigating, and leads him, by means of measurements which can be independently verified, to that knowledge of his powers which is so important when he has to do any original work. Since the greater part of the treatise is devoted to measurements of physical quantities, we have thought its object better expressed by the title we have given it than by a literal translation of the German one.

Descriptions of apparatus are but rarely given, as students mostly have instruments provided for them, and seldom have to make their own apparatus, or to put it together.

The mathematical knowledge required is but very elementary, as the proofs of the formulæ are only given when they present no complex arguments.

The present edition is translated from the seventh German edition, published in 1892, in the preface to which Dr. Kohlrausch explains that the scope of the work has been somewhat enlarged, so as to include not only the necessary

practice for students, but also to some extent explanations and directions referring more definitely to investigation and research. This widening of aim has naturally produced a considerable increase of matter, and the present edition contains nearly four times the number of pages that the first (German) one did in 1869.

From the same cause the bibliography has been considerably extended, regard being always had, as the author is careful to state, to the accessibility and serviceable character of the reference, without any expression of judgment as to priority.

Dr. Kohlrausch has also obtained for special sections of the work the collaboration of distinguished specialists, among whom we may mention Pfaundler, Dorn, Hallwachs, Neumayer, and W. Kohlrausch.

A good deal of new matter has been embodied in the tables, which have also been corrected to the present state of knowledge.

For the sake of facilitating the use of this edition, where the former ones have been employed the arrangement and numbering of the sections have been as nearly as possible retained.

Some notes, appendices, and tables, signed "TR.," have been added by the translators, for which they alone are responsible.

August 1894.

CONTENTS

Introductory

Weight and Density

Elasticity and Sound

Capillarity and Friction

Light

Methods of Observation frequently employed in Magnetic and Electrical Work

Magnetism

GALVANISM

AN INTRODUCTION

TO

PHYSICAL MEASUREMENTS

AN INTRODUCTION

TO

PHYSICAL MEASUREMENTS

PHYSICAL MEASUREMENTS

1.—Errors of Observation. Mean and Probable Error.

To measure a quantity is to express it by a number which states how often the suitably chosen unit is contained in the quantity.

The numerical value of a physical quantity is affected with error from the inaccuracy of the observation. If the same quantity have been repeatedly measured, we require some means of calculating the most probable value in order to obtain, from the amount of agreement of the observations, an opinion as to the probable limits of error.

When all the separate determinations are, in the opinion of the observer, entitled to an equal degree of confidence, the arithmetical mean of the separate determinations gives, as is well known, the most probable value of the required quantity, —that is, all the separate values are added together, and the sum divided by the number of determinations.

We may here insist upon the fact that it is generally quite inadmissible arbitrarily to exclude from a series of observations some of the number, simply because they do not agree with the greater number. The probability of an increased error being introduced by the irregular numbers will be compensated by the very process of taking the arithmetical mean, for as single ones among a greater number they have a small influence upon the mean value.

If now the separate determinations be compared with the mean value, there will be found greater or less differences, the "errors" from the amount of which the *probable error* of an

observation as well as that of the result can be found by the following rules. First, the sum is taken of the *squares of the errors.* The sum divided by the number of observations diminished by 1 gives the square of the mean error; the square root of it is the *mean error of a single observation.* If now this mean error be divided by the square root of the number of observations, we obtain what is called the *mean error of the result.*

Multiplying the mean error by 0·674 (or $\frac{27}{40}$, or with sufficient accuracy for most purposes by $\frac{2}{3}$), we get the *probable error.* This last expression means that it is as likely that the actual unknown error is less than the "probable error" as it is that it is greater. By prefixing the sign $\pm$ we indicate that the mean value is as likely to be too great as it is to be too little within the limits of the probable error.

Let us then call

n the number of observations;
$\delta_1, \delta_2, \delta_3 \ldots \delta_n$ the deviations from the arithmetical mean;
S the sum of the squares of the errors; *i.e.*

$$S = \delta_1^2 + \delta_2^2 + \delta_3^2 + \ldots \delta_n^2;$$

then the mean error of a single observation $\epsilon = \pm\sqrt{\frac{S}{n-1}}$; the mean error of the result obtained by taking the arithmetical mean

$$E = \pm\sqrt{\frac{S}{n(n-1)}}$$

the *probable* errors amount to $\frac{2}{3}$ of these.

On the calculation of errors with several unknown quantities see **3**.

It will be obvious that only that part of the error is expressed by quantities thus calculated, which is introduced by true uncertainty of observation—that is, by such errors as give too great a value as often as too small a value. But there may exist *constant errors,* the cause of which may be in the indications of the instrument, or which may be so related to them that the observer makes errors which preponderate in one definite direction. It is an important problem either to find out such errors and then correct the result, or to make such

combinations of the results or such changes of method that the constant errors are thereby eliminated.

Example.—The density of a body was determined ten times, with the results given in the first column.

	Found.	Difference δ from the Mean.	δ^2
	9·662	− 0·0019	0·000004
	9·673	+ 091	083
	9·664	+ 001	000
	9·659	− 049	024
	9·677	+ 131	172
	9·662	− 019	004
	9·663	− 009	001
	9·680	+ 161	259
	9·645	− 189	357
	9·654	− 0·0099	0·000098
Mean	9·6639		$S = 0{\cdot}001002$

Then since $n = 10$ we have

the mean error of one observation $= \sqrt{\dfrac{0{\cdot}001002}{10-1}} = \pm 0{\cdot}011$

the mean error of the result $= \sqrt{\dfrac{0{\cdot}001002}{10 \times 9}} = \pm 0{\cdot}0033$

probable error of one observation $= 0{\cdot}674 \sqrt{\dfrac{0{\cdot}001002}{9}} = \pm 0{\cdot}0071$

probable error of the result $= 0{\cdot}674 \sqrt{\dfrac{0{\cdot}001002}{10 \times 9}} = \pm 0{\cdot}0023$

According to this we may wager one to one that the error which affects the separate determinations of the density of this body, with the instruments, care, and experience supposed above, is less than 0·0071. It happens accidentally that just half the above differences are smaller, the other half greater, than this amount.

The probable error deduced from a series of only 10 observations can only be considered as an approximation. It was really superfluous to calculate it out to three places as we have done. Similarly the approximate value $\frac{2}{3}$ might have been used instead of 0·674.

The determinations given above were made by different observers, using different sets of weights and different thermometers. Errors of the balance, which affect the determination of the density in one direction only, are not in question. An error with effect in one direction only, though not a constant one, might however arise, for example, by the buoyancy of air-bubbles not completely removed from the body when weighed in water, for from this cause the density must always be found too small.

"Weight" of an Observation.—The separate results from which the final result is calculated are not always equally reliable. We seek to take account of this circumstance by assigning to each separate result a different "weight," that is, in taking the mean of the observations, by multiplying by 2 or 3, etc. (weight 2 or 3, etc.). This is, of course, only done where the separate results are already deduced from several observations. The "weight" is then simply set equal to the number of observations. For by calculating in this manner the end result is the same as if the mean had been taken of all the individual results. There may be other causes also which produce a different degree of certainty in the results, and lead to the assigning of different weights; the decision of this question must be left to the judgment of the observer.

If the mean error, ϵ, of a result is known, the weight p is proportional to $\frac{1}{\epsilon^2}$; see also **3**, III. The arithmetical mean of several results r_1, r_2, etc., of weights p_1, p_2, etc., is naturally

$$r = \frac{p_1 r_1 + p_2 r_2 + \text{etc.}}{p_1 + p_2 + \text{etc.}}$$

2.—Influence of Errors of Observation on the Result.

We frequently do not find a result directly by observation, but must deduce it from an observed magnitude, or even from several such, by calculation. Thus the density of a body is calculated from several weighings, the modulus of elasticity from measurings of length, the strength of a galvanic current from

the deflection of a needle, according to certain formulæ. Hence arises the problem to determine to how great an extent the result will be in error when the observed magnitudes are affected by a certain error.

The object of this calculation of errors may sometimes be to form a judgment as to the accuracy of the result itself. Further, we learn from it what abbreviation of the calculation we may allow ourselves without unduly increasing the inaccuracy. In cases where the measurement is the result of several observations, it also shows us over what part we must expend the greatest care. Finally, it is frequently in our power to vary the proportions of the experiment in different ways: this calculation of errors alone gives us the information as to what choice of ratio is most advantageous, *i.e.* which gives the least influence upon the result to errors of observation.

Such are, for instance, the considerations from which the rule given on p. 243 is derived—that in determining the horizontal intensity of the earth's magnetism, it is best to take the distances of the deflecting magnet in the ratio 4 : 3. In the same way also are got the rules, that the measurement of the strength of a galvanic current with a tangent galvanometer furnishes the most accurate results with an angle of deflection of about 45°; that the two current strengths, from which the resistance (p. 322) or the electromotive force of a galvanic battery are determined are most advantageously in the ratio 1 : 2, that the damping ratio of an oscillating body is most accurately observed when the ratio of the two elongations is 2·8, etc.

If we call the observed magnitude x, and the required result X, where x and X represent the correct values, X will be some function of x—*i.e.* will be given by some mathematical expression in which x occurs. If now we call f the error of x, the error introduced by it into X which we call F, is found by putting $x+f$ instead of x in the expression from which X is calculated. We shall now have a result somewhat different from X, the correct value; the magnitude of this difference is manifestly the error F.

Since the errors of observation are small quantities, this calculation may be much simplified. We first note the following rules:—

1. In determining the errors in the result, it is sufficient to use an approximate value for the observed magnitude, which we have called x. Indeed we are always compelled to do so, since the true, accurate value is not known.
2. Correction terms (4) which occur in the formula for the result X, may, if we are not inquiring into their influence, be neglected in calculating the error.
3. When a measurement depends on several independent observations, the final result will be an expression compounded of the separately observed quantities. Several of these may be affected by errors. But if the influence of the errors introduced by one of the magnitudes is to be determined, the others need not be taken any account of.
4. The error in the result which arises from an error of observation varies proportionally with this latter. In other words, the difference which we have above called F may be represented as a product of which the error f of the observed magnitude is one factor.
5. From this it follows also that the errors of the result which arise from errors of observation, equal in magnitude but opposite in sign, are also equal in magnitude but have contrary signs.

It occasionally happens that the error of the result is not proportional to the error of observation, but, for instance, to its square, or to the product of several errors. In such cases rules 4 and 5, and occasionally 3, are of course inapplicable.

The calculation may almost always be made very much shorter by the use of approximation formulæ for calculating with small magnitudes. These may easily be constructed by the aid of the differential calculus. If f be the error which occurs in the observed value, the error F of the result X is obtained by multiplying the partial differential coefficient of X with regard to x by f. Therefore

$$\mathrm{F} = f\frac{dX}{dx}$$

In order to bring the expression for the error to a simple form, without the use of the differential calculus, it will very often, if not always, be possible to adopt the plan for the calculation of correction quantities given at the end of this article: by suitable transformations we must arrange that the error of observation f occurs only as a small quantity added to or subtracted from 1, upon which for further reduction the formulæ given below, or special ones, may be at once used.

When the result has been got from several observations combined, we may, according to No. 3 (see p. 6), investigate the influence of the single errors separately. Each of them may of course make the result either too small or too great, and the total error will be larger or smaller according as the signs happen to be the same or different. The maximum of error will be obtained when the partial errors have the same sign. The error *probably* arising is found by adding the squares of the partial errors, and taking the square root of the sum. The employment of these rules in a special case will serve to explain this sufficiently.

We choose as our example the determination of the density of a solid body which sinks in water, by the ordinary method, in which the body is weighed in air and in water. We will determine the effect of an error in weighing upon the density deduced from this weighing. If we call the weight of the body in the air m, and the weight in water m', the density is

$$\frac{m}{m-m'}$$

To this formula must of course be added the corrections depending upon the loss of weight in the air, and upon the expansion of the water; but according to No. 2, p. 6, we need not trouble with these in the simple calculation of the error.

According to No. 3 we may consider the errors in m and m' separately, since they are independent of one another. Let us therefore find first the influence upon the result of an error in the weight in air. If we had committed the error f in this weighing, we should, instead of the true weight m, have found $m+f$, and should therefore obtain the density $\dfrac{m+f}{m+f-m'}$.

Using formula 8, p. 11, we will write for this

$$\frac{m}{m-m'} \cdot \frac{1+\frac{f}{m}}{1+\frac{f}{m-m'}} = \frac{m}{m-m'}\left(1+\frac{f}{m}-\frac{f}{m-m'}\right) = \frac{m}{m-m'}, -f\frac{m'}{(m-m')^2}$$

The first term of the last expression is, however, the true result; so that

$$F = -f\frac{m'}{(m-m')^2}$$

is the error produced by the error $+f$ in weighing the body in air.
The differential calculus gives at once the same result,

$$F = f\frac{d\frac{m}{m-m'}}{dm} = -f\frac{m'}{(m-m')^2}$$

Secondly, let us consider an error committed in the weighing in water, which we will call f'. Setting therefore $m'+f'$ instead of m', the result affected with the error will be, as above,

$$\frac{m}{m-(m'+f')} = \frac{m}{m-m'-f'} = \frac{m}{(m-m')\left(1-\frac{f'}{m-m'}\right)} =$$

$$\frac{m}{m-m'}\left(1+\frac{f'}{m-m'}\right) = \frac{m}{m-m'} + f'\frac{m}{(m-m')^2}$$

That is to say, by observing the weight in water as too great by f', we shall make the result too great by $F' = f'\frac{m}{(m-m')^2}$

If, finally, we inquire as to the total error, which is compounded of the two errors of observation f and f', this has obviously its maximum value $\pm\frac{m'f+mf'}{(m-m')^2}$ when either m was found too great and m' too small or *vice versâ*. The *probable* total error is

$$\pm\sqrt{F^2+F'^2} = \pm\frac{\sqrt{(m'f)^2+(mf')^2}}{(m-m')^2}$$

We will take, as a numerical example, the determination of the density of the same body of which we have already spoken, p. 7. We have there determined the amount of the error by the difference of the results which we obtained from their mean value. We want now to see what amount of error is to be expected from inaccurate observation in the weighing.

The weight of the piece was, in round numbers,

In air = 243,600 mgrs.
In water = 218,400 mgrs.

The greatest error in weighing, with the balance made use of, with moderate care, for loads such as the above, may be reckoned at 5 mgrs. when weighing in the air, at 8 mgrs. when weighing in water; which latter operation, on account of the friction of the water, is less accurate, whence

$$f = 5 \text{ mgrs.} \quad f' = 8 \text{ mgrs.}$$

(The errors must be reckoned in the same units as the observed weights themselves.)

The stated quantities substituted in the formulæ given above give,

as the error depending on m, $\pm \dfrac{5 \times 218400}{25200^2} = \pm 0{\cdot}0017 = F$

" " m', $\pm \dfrac{8 \times 243600}{25200^2} = \pm 0{\cdot}0031 = F'$

In the most unfavourable case the total error amounts to 0·0048, but in the most probable case $= \pm \sqrt{F^2 + F'^2} = \pm 0{\cdot}0035$.

As, therefore, single ones of the above given determinations give considerably greater differences, there must have been present other sources of error besides the uncertainty of the weighing—(bubbles of air, inaccuracy in determining the temperature, mistakes in reckoning up the weights).

As a second example, the measurement of the strength of a galvanic current i with the tangent galvanometer may serve. If ϕ be the angle of deflection of the needle, we have

$$i = C \tan \phi,$$

where C is a factor constant for the same instrument. If an error f occur in the reading off of the angle ϕ, the error F in i follows from

$$i + F = C \tan (\phi + f),$$

or by formula 10 (p. 11),

$$i + F = C \left(\tan \phi + \frac{f}{\cos^2 \phi}\right); \text{ therefore}$$

$$F = C\frac{f}{\cos^2 \phi} = i\frac{f}{\sin \phi \,.\, \cos \phi} = i\frac{2f}{\sin 2\phi}$$

$\dfrac{2f}{\sin 2\phi}$ is therefore the error, expressed as a fraction of i, which corresponds to an error f in the reading off of the deflection. Hence we have the very important rule for the use of the tangent galvanometer—that angles of about 45° are most advisable for the accuracy of the measurement, because the denominator $\sin 2\phi$ has its maximum value (viz. 1) when $\phi = 45°$.

Rules for Approximation when Calculating with Small Quantities.

When, in a mathematical expression, some numbers are very small in comparison with others, the expression may often be brought into a form more convenient for calculation by the use of formulæ of approximation. It will very frequently recommend itself as the simplest to first give the expression such a form that the corrections are contained in terms added to or subtracted from 1, and very small compared with 1; this is not unfrequently the form in which it is already given. It will then often be possible to make use of one of the following formulæ to simplify the expression.

In these formulæ let the magnitudes denoted by δ, ϵ, ζ . . . be very small compared with 1, so small that their second and higher powers δ^2, ϵ^2 . . . as well as their products $\delta\epsilon$, $\delta\zeta$ . . . which, again, are very small compared with δ, ϵ . . . themselves, may practically be completely neglected compared with 1.

If, for example, $\delta = 0{\cdot}001$, $\delta^2 = 0{\cdot}000001$; if further, $\epsilon = 0{\cdot}005$, $\delta\epsilon = 0{\cdot}00005$;—it often happens that things which affect a quantity to the extent of some thousandths are important, whilst some millionths more or less are a matter of complete indifference. It is usually easy to measure a length of 1 meter accurately to the tenth of a millimeter. It would not do, therefore, to neglect a correction of a thousandth of the length, or 1 mm. But one, or several, millionths of the total length—*i.e.* thousandths of a millimeter—will most rarely have any practical influence, since the errors of observation are much greater.

On this supposition it may be easily shown that the following formulæ hold good, in which the expressions to the right of the sign of equality will usually be more convenient for calculation.

Where the sign $\pm$ or $\mp$ is placed before a quantity, either the upper or lower sign must be taken all through the formula

$$(1+\delta)^m = 1 + m\delta. \qquad (1-\delta)^m = 1 - m\delta. \tag{1}$$

therefore in different cases

$$(1+\delta)^2 = 1 + 2\delta. \qquad (1-\delta)^2 = 1 - 2\delta. \tag{2}$$

$$\sqrt{1+\delta} = 1 + \tfrac{1}{2}\delta. \qquad \sqrt{1-\delta} = 1 - \tfrac{1}{2}\delta. \tag{3}$$

$$\frac{1}{1+\delta} = 1 - \delta. \qquad \frac{1}{1-\delta} = 1 + \delta. \tag{4}$$

$$\frac{1}{(1+\delta)^2} = 1 - 2\delta. \qquad \frac{1}{(1-\delta)^2} = 1 + 2\delta. \tag{5}$$

$$\frac{1}{\sqrt{1+\delta}} = 1 - \tfrac{1}{2}\delta. \qquad \frac{1}{\sqrt{1-\delta}} = 1 + \tfrac{1}{2}\delta, \text{ etc.} \tag{6}$$

$$(1 \pm \delta)(1 \pm \epsilon)(1 \pm \zeta)\ldots = 1 \pm \delta \pm \epsilon \pm \zeta \ldots \tag{7}$$

$$\frac{(1 \pm \delta)(1 \pm \zeta)\ldots}{(1 \pm \epsilon)(1 \pm \eta)\ldots} = 1 \pm \delta \pm \zeta \ldots \mp \epsilon \mp \eta \ldots \tag{8}$$

Thus also we may, instead of the geometrical mean of two quantities p_1 and p_2, only slightly different from each other, use the arithmetical mean

$$\sqrt{p_1 p_2} = \frac{p_1 + p_2}{2} \tag{9}$$

Further,

$$\begin{aligned} &\sin(x+\delta) = \sin x + \delta \cos x; && \sin \delta = \delta \\ &\cos(x+\delta) = \cos x - \delta \sin x; && \cos \delta = 1 \\ &\tan(x+\delta) = \tan x + \frac{\delta}{\cos^2 x}; && \tan \delta = \delta \end{aligned} \tag{10}$$

in which δ signifies a small angle measured in terms of the angle ($57^\circ{\cdot}3$), for which the arc is equal to the radius.

As a second approximation

$$\sin \delta = \delta(1 - \tfrac{1}{6}\delta^2); \ \cos \delta = 1 - \tfrac{1}{2}\delta^2; \ \tan \delta = \delta(1 + \tfrac{1}{3}\delta^2) \tag{11}$$

Finally,

$$\text{nat log}\,(x+\delta) = \text{nat log}\, x + \frac{\delta}{x}; \ \text{nat log}\,(1+\delta) = \delta \tag{12}$$

3.—Determination of Empirical Constants by the Method of Least Squares.

I. If the same magnitude has been measured several times, the arithmetical mean gives the most probable value. But frequently the required magnitude is not the immediate object of the measurement, but must be deduced, by calculation, from the observations, according to known physical laws, and then the arithmetical mean is not always sufficient to find the most probable result from repeated measurements.

Mathematically considered, the quantity sought occurs here as a constant in an equation which also contains the observed magnitudes. Not infrequently other unknown con-

stants occur in this equation, and are to be at the same time determined, or at least eliminated. For this purpose at least as many observations are required as there are unknown quantities; and if there be only just as many, we must, by substituting the observed values in the mathematical expression, make as many equations as there are unknown quantities, and deduce the latter from them in the ordinary way (see also III., p. 17). But when a large number of observations has been made, we must, in order to utilise all the materials, employ another method—of which the use may be facilitated by various devices, especially by adapting the observations to a plan determined beforehand.

The calculation of probabilities by the method of least squares affords a systematic course of proceeding by which the calculations may be made without bias. Of course it may frequently be found that by this method we are also led into tiresome calculations, which form another proof of the advantage afforded by a plan completely thought out before the observations are made.

As an example we take the simple problem of determining the length of a rod at 0°, and its expansion for 1° of temperature, from a number of measurements at different temperatures. If we call the length at $0° = a$, and the expansion for $1° = b$, we have for the length y, at any temperature x

$$y = a + bx$$

a and b are two unknown constants, for determining which two observations would be sufficient. Suppose, for example, we had observed the lengths y_1, y_2, at the temperatures x_1, x_2, respectively, we should have

$$y_1 = a + bx_1, \quad y_2 = a + bx_2$$

therefore

$$a = \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2}, \quad b = \frac{y_1 - y_2}{x_1 - x_2}$$

But more than two observations may have been made; suppose besides the pairs given above, x_3, y_3, x_4, y_4, etc. If the observations were free from error, the quantities sought, a and b, would have the same numerical value when calculated from any two pairs; and, on the other hand, every value of y, calculated by this formula from the corresponding value of x, would be identical with the

observed value. But, in reality, we find that on account of errors no values for a and b completely satisfy *all* the observations.

The fundamental law of the method of least squares is: The constants must be so chosen that the sum of the squares of the errors is a minimum. That is to say, with every different value of the constants the values calculated from the law by means of them will differ from the observed values by different amounts (the errors). The most probable values of the constants are found when the sum of the second powers of all the differences is the smallest possible number.

If we denote the mathematical expression of known form, which gives the dependence of the observed magnitude y, on another, x (or on several others), by the general expression $f(x)$, the magnitudes we seek occur in it as constants which we call a, b .... Our equation then is

$$y = f(x)$$

Let several values y_1, y_2, y_3 .... be observed corresponding to the known values x_1, x_2, x_3 .... By the above law the numerical values of a, b .... are to be so determined that when they are substituted in $f(x)$, the sum of the squares of the differences between the calculated and observed values has the smallest value possible. Therefore we must have

$$\{y_1 - f(x_1)\}^2 + \{y_2 - f(x_2)\}^2 + \{y_3 - f(x_3)\}^2 + .. + \{y_n - f(x_n)\}^2 = \text{a minimum},$$

or, introducing the symbol of summation,

$$\Sigma \{y - f(x)\}^2 = \text{a minimum}.$$

We must keep in mind that all the values of x and y are known, observed quantities. For the methods by which in case of necessity the equations are previously brought to the same degree of accuracy, see IV.

By a law of the differential calculus, this condition produces as many equations as there are quantities a, b ... to be determined. We differentiate the expression $\Sigma\{y - f(x)\}^2$ with respect to a, b ... considering these as the variables, and equate each partial differential coefficient to zero.

The equations from which $a, b \ldots$ are to be determined become therefore

$$\frac{d\Sigma\{y - f(x)\}^2}{da} = 0, \quad \frac{d\Sigma\{y - f(x)\}^2}{db} = 0, \text{ and so on.}$$

We have thus found a way, free from any uncertainty, by which we can make equal use of as many observations as we please.

Of course it may happen, with complicated forms of $f(x)$, that the equations derived by differentiation with respect to $a, b \ldots$ are not capable of direct solution. In such cases we must find a solution by trial and approximation. In the important case, however, where $f(x)$ has the form, $f(x) = a + bx + cx^2 + dx^3 + \ldots$ the direct solution is always possible. See also III. and IV.

Let us illustrate the problem by the example given above. Let the lengths of the rod observed at $x_1, x_2, x_3 \ldots x_n$, be $y_1\, y_2\, y_3 \ldots y_n$. According to the law of expansion with temperature $y = a + bx$, and so what we have above called $f(x)$ is here $f(x) = a + bx$. We have therefore to determine a and b, so that $(y_1 - a - bx_1)^2 + (y_2 - a - bx_2)^2 + \ldots + (y_n - a - bx_n)^2$ = a minimum, or briefly $\Sigma(y - a - bx)^2$ = a minimum.

Differentiation gives

$$\begin{aligned} &\text{with respect to } a, \quad \Sigma(y - a - bx) = 0 \\ &\text{with respect to } b, \quad \Sigma x(y - a - bx) = 0 \end{aligned}$$

or observing that with n observations $\Sigma a = an$,

$$\begin{aligned} \Sigma y - an - b\Sigma x &= 0 \\ \Sigma xy - a\Sigma x - b\Sigma x^2 &= 0 \end{aligned}$$

By solving these equations with respect to a and b, we have

$$a = \frac{\Sigma x\, \Sigma xy - \Sigma y\, \Sigma x^2}{(\Sigma x)^2 - n\Sigma x^2}$$

$$b = \frac{\Sigma x\, \Sigma y - n\Sigma xy}{(\Sigma x)^2 - n\Sigma x^2}$$

As an example, suppose the length of a measuring rod, which is to be corrected by comparison with a normal scale.

At temperature (x) =	20°	40°	50°	60°
the length has been found =	1000·22	1000·65	1000·90	1001·05 mm.

In order to shorten the calculations we take as y only the observed excesses of the length above 1000 mm. We shall then have for a the excess of the length at 0° above 1 meter.

The calculation is performed as follows :—

x	y	x^2	xy
20	+0·22	400	4·4
40	0·65	1600	26·0
50	0·90	2500	45·0
60	1·05	3600	63·0
$\Sigma x = 170$	$\Sigma y = 2{\cdot}82$	$\Sigma x^2 = 8100$	$\Sigma xy = 138{\cdot}4$

$$\text{therefore } a = \frac{170 \,.\, 138{\cdot}4 - 2{\cdot}82 \,.\, 8100}{170^2 - 4 \,.\, 8100} = -0{\cdot}196 \text{ mm.}$$

$$b = \frac{170 \,.\, 2{\cdot}82 - 4 \,.\, 138{\cdot}4}{170^2 - 4 \,.\, 8100} = +0{\cdot}0212 \text{ mm.}$$

The length of the rod at 0° is therefore 999·804 mm. and at the temperature t, $999{\cdot}804 + 0{\cdot}0212\ t$.

If now the lengths are calculated for 20°, 40°, 50°, 60°, we shall find—

x	y Calculated mm.	Observed mm.	Error Δ mm.	Δ^2
20°	1000·228	1000·22	+0·008	0·000064
40	1000·652	0·65	+0·002	0004
50	1000·864	0·90	−0·036	1296
60	1001·076	1·05	−0·026	0676
				$\Sigma\Delta^2 = 0{\cdot}002040$

The student may verify that any alteration of a or of b increases the sum of the squares of the errors.

Exactly the same method of proceeding would be employed to find the modulus of elasticity of a rod, or to determine the relative rate of two clocks.

For the expansion of fluids with temperature, and in many similar cases, it is usual to use as approximations an algebraical formula of a higher degree, e.g. $y = a + bx + cx^2$. The determination of a, b, c from any number of observations is essentially the same as above, only more complicated and tiresome.

In such cases, and when the method of least squares has

often to be used, even in simple cases, Gauss's method of calculation (IV.) is more convenient and safe.

With regard to the numerical calculations the following practical rules should be observed. The constants a, b, and c, as well as the observed values, are often of various orders of magnitude. Thus in the example given above the temperatures are counted by tens, while the elongations (y) reach at greatest 1 mm. It is more convenient to have the magnitudes similar in amount, which may be brought about by multiplying or dividing by powers of 10. Instead of bx we may, for instance, write $10b \times \frac{x}{10}$. Had this been done we should have had 2, 4, etc., instead of 20, 40, and the calculation would have been easier. The result $10b$ would then have to be divided by 10.

The so-called *mean error of an observation* is obtained in this method from the sum of the squares of the differences between observed and calculated magnitudes, if n = number of observations, m that of the constants $a, b, c \ldots$ to be determined by the formula

$$\pm\sqrt{\frac{\Delta_1^2 + \Delta_2^2 + \ldots \Delta_n^2}{n - m}}$$

Therefore in the above example, where $n = 4$, $m = 2$, we have

$$\pm\sqrt{\frac{0\cdot00204}{4 - 2}} = \pm\, 0\cdot032 \text{ mm.}$$

II. Calculation by Equal Intervals.

If the observed quantities are separated by equal intervals, the calculation is simplified. This not infrequently occurs; for instance, when a periodic phenomenon is observed, and the time between two consecutive occurrences is determined [time of oscillation or rotation (**52**)]; or when the distance between pairs of points is to be determined, of which many occur consecutively, and of which the places can be measured on a measuring rod [distance of the nodal points of waves (**37**)].

Taken more generally, if one quantity vary proportionally

to another, a number of points separated by equal intervals must be taken in the variation of the latter, and the values of the first quantity which correspond to them must be observed.

So in the previous example the lengths of the bar might be measured at equal intervals of temperature.

The values of the observed quantity y may be thus found as a series $y_1, y_2 \dots y_{n-1}, y_n$. If these values were accurately observed, the intervals $y_2 - y_1, y_3 - y_2 \dots y_n - y_{n-1}$, should be equally great. Actually, they are unequally great, and it is their most probable value which we seek. To take their arithmetical mean would obviously amount to the same as if we noted only the first and last values, and neglected all the intermediate ones. To utilise all the observations, it is necessary that we should calculate the interval as

$$6\frac{(n-1)(y_n - y_1) + (n-3)(y_{n-1} - y_2) + \dots}{n(n^2-1)}$$

The weight of the result thus obtained is $P = \frac{n(n^2-1)}{12}$; the mean error of the result amounts, when e is the mean error of the single observation, to $E = \frac{e}{\sqrt{P}}$.

Let t signify the number of the observation, and $y = a + bt$, then b is the interval sought. Therefore

$$t_1 = 1,\ t_2 = 2 \dots t_{n-1} = n-1,\ t_n = n.$$

If, then, in the value given for b (p. 14), the following be substituted, the formula above given follows:

$$\begin{aligned} \Sigma t &= 1 + 2 + \dots + n = \tfrac{1}{2}n(n+1) \\ \Sigma t^2 &= 1^2 + 2^2 + \dots + n^2 = \tfrac{1}{6}n(n+1)(2n+1) \\ \Sigma y &= y_1 + y_2 + \quad y_n \\ \Sigma ty &= y_1 + 2y_2 + \quad ny_n \end{aligned}$$

III. Solution of Equations, when Approximations for the Unknown Quantities can be assumed.

We will first show how the determination of the constants of an equation may be made to depend upon the solution of linear equations.

Let an observed magnitude u be connected with other observed magnitudes r, s, t, by a law of known form. This law contains the constants A, B, C, which are to be determined from the observations. In physical problems a larger number than 3 seldom occurs, but such cases would be treated exactly as our example. There must of course be at least as many observations as constants to be determined. The observed magnitudes are denoted by $u_1r_1s_1t_1$, $u_2r_2s_2t_2$, etc., and the dependence of the magnitudes on each other by

$$u = f(A, B, C, r, s, t)$$

It will generally be the case that r, s, t, and frequently also u are readings of instruments such as clock, balance, measuring rod, divided circle, thermometer, manometer, rheostat, galvanometer, etc., r, s, t need not be independent of each other. A common case, for instance, is that in which $r = 1$, $s = q$, $t = q^2$, where q may be a temperature, load, time, pressure, scale reading, etc. Or, if u is the deflection of a magnetometer by a magnet at distance l, we may have $r = l^{-3}$, $s = l^{-5}$; while A may then signify the magnetism of a bar, and B the distance apart of its poles, u might also represent a density, an electric current or resistance, a refractive index, etc., etc.

The equations are frequently incapable of a direct solution. If, however, approximate values of the required constants A, B, C have been obtained, the problem can be reduced in the following manner to the solution of linear equations which is always possible.

Let the approximate values be $A_0B_0C_0$ the true values will then be $A = A_0 + a$, $B = B_0 + \beta$, $C = C_0 + \gamma$. These corrections a, β, γ are therefore now the unknown quantities, and are to be determined. For this purpose, take the partial differential coefficients of the function u with regard to A, B, C, considering these for the time being as the variables, but after differentiation substituting for them $A_0B_0C_0$.

We call the values so defined of the differential coefficients

$$\frac{du}{dA} = a, \quad \frac{du}{dB} = b, \quad \frac{du}{dC} = c$$

In these three expressions the observed numbers must be substituted for $r, s, t \ldots$, e.g. $r_1, s_1, t_1 \ldots$, then $r_2, s_2, t_2 \ldots$, etc. The magnitudes thus obtained are represented by $a_1, b_1, c_1 \ldots$, $a_2, b_2, c_2 \ldots$, etc.

Finally, we call u_0 the value obtained for the function u when $A_0 B_0 C_0$ are substituted in the expression f, whilst u is the true actually observed value. u and u_0 differ from each other by a quantity $\mathfrak{r}$, so that

$$\mathfrak{r} = u - u_0$$

According to Taylor's theorem, if $\mathfrak{r}, \alpha, \beta, \gamma$ are small enough

$$\mathfrak{r} = \alpha \frac{du}{dA} + \beta \frac{du}{dB} + \gamma \frac{du}{dC} = \alpha a + \beta b + \gamma c$$

By substituting the observations in these equations, as many equations as observations are obtained, in which, except α, β, γ, everything is given numerically.

$$\mathfrak{r}_1 = \alpha a_1 + \beta b_1 + \gamma c_1$$
$$\mathfrak{r}_2 = \alpha a_2 + \beta b_2 + \gamma c_2$$
etc., etc.

Example.

Let the temperature of a body cooling in constant conditions of its surroundings (or the position of a "damped" magnet needle, or of a body moving in a viscid medium, or the progress of a slowly proceeding chemical reaction, etc.) be for the time t given by the expression

$$u = A \,.\, 10^{-Bt} + C$$

Let approximate values A_0, B_0, and C_0 have been obtained by the combination of three observations. We have then

$$\mathfrak{r} = u - A_0 10^{-B_0 t} - C_0$$

$$a = \frac{du}{dA} = 10^{-B_0 t}, \quad b = \frac{du}{dB} = -\frac{A_0 t_0 10^{-B_0 t}}{\log e}, \quad c = \frac{du}{dC} = 1$$

If there are only as many observations as unknown quantities the equations can be solved in the ordinary manner; in other cases the method of least squares may be used (see I. and IV.)

It is scarcely necessary to remark that, where u is already

given by the formula $u = Ar + Bs + Ct$ (e.g. $u = A + Bs + Cs^2$, where s represents, for instance, a temperature) the mutual reductions are not necessary. Nevertheless, a similar method may frequently be advantageous, viz. to obtain approximations for A, B, C, and to perform the calculations with the remainders, so that smaller numbers may be introduced which can be calculated mentally or with tables. In this case—

$$a = \frac{du}{dA} = r, \quad b = \frac{du}{dB} = s, \quad c = \frac{du}{dC} = t$$

This is obvious without employing the differential calculus, for if

$$u_0 + \mathfrak{r} = (A_0 + a)r + (B + \beta)s + (C + \gamma)t$$
$$\mathfrak{r} = ar + \beta s + \gamma t$$

The advantage gained will be seen from the example on p. 14. The formula was $u = A + Bt$. An approximate value for B is plainly obtained from the two observations $t_1 = 20$, $u_1 = 1000{\cdot}22$, and $t_4 = 60$, $u_4 = 1001{\cdot}05$, viz.—

$$B_0 = \frac{1001{\cdot}05 - 1000{\cdot}22}{60 - 20} = \frac{0{\cdot}83}{40} = 0{\cdot}021.$$

From observation (1) an approximation is then found for A.

$$A_0 = 1000{\cdot}22 - 20 \times 0{\cdot}021 = 999{\cdot}8$$

therefore

$$u_0 = 999{\cdot}8 + 0{\cdot}021t$$

since now

$$a = \frac{du}{dA} = 1 \quad b = \frac{du}{dB} = t$$

the equation becomes

$$u - u_0 = \mathfrak{r} = a \,.\, 1 + \beta t$$

u mm.	t	$u_0 = 999{\cdot}8 + 0{\cdot}021t$ mm.	$\mathfrak{r}$ mm.	a	b
1000·22	20	1000·22	± 0·00	1	20
1000·65	40	1000·64	+ 0·01	1	40
1000·90	50	1000·85	+ 0·05	1	50
1001·05	60	1001·06	− 0·01	1	60

where now all can be mentally calculated; most simply by expressing $\mathfrak{r}$ in hundredths of a millimeter, and b in 10 degrees as units, and finally dividing the calculated values of a and β by 100 and 1000 respectively.

IV. Gauss's Method for the Solution of Linear Equations using Least Squares.

Let there have been n observations made, and, in case of need, brought into a linear form with approximate values in the way given in III. Let the equations to be solved for a, β, γ be:

$$\begin{aligned} r_1 &= a a_1 + \beta b_1 + \gamma c_1 \\ r_2 &= a a_2 + \beta b_2 + \gamma c_2 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ r_n &= a a_n + \beta b_n + \gamma c_n \end{aligned}$$

It will usually suffice to calculate out $a_1 \ldots b_1 \ldots c_1 \ldots$ so far that the number of figures about equals that of the remainders r. It is a great advantage when the approximate values are so nearly true that the remainders contain two or at the most three figures. The squares and products can then be taken from tables or worked with logarithms. In these calculations, as already stated, no further abbreviations should be made.

All equations should be of the same probable degree of accuracy. If there is any reason for ascribing a different degree of accuracy to the different observations, the equations must be reduced to equality in this respect by multiplication by the square root of the weight proper to each.

The procedure of p. 13 may serve to solve the equations according to the method of least squares, noticing that the expressions

$$y, x \ldots, AB \ldots$$

correspond to the

$$r, abc \ldots, a\beta\gamma \ldots$$

in this section.

For a larger number of unknown quantities, however, indeed even for our three, the following solution is more convenient.

On the left hand is the series of main calculations; on the right a check on their accuracy which is worked out along with the main calculation. First, the sums of the squares or the products of corresponding magnitudes are to be calculated, which are denoted in the following obvious abbreviated manner:—

$$a_1^2 + a_2^2 + \ldots, a_n^2 = \overline{aa};\ a_1 b_1 + a_2 b_2 + \ldots a_n b_n = \overline{ab}$$
$$a_1 r_1 + a_2 r_2 + \ldots + a_n r_n = \overline{ar}, \text{ etc.}$$

Calculation.

We require

$$\begin{array}{cccc} \overline{aa} & \overline{ab} & \overline{ac} & \overline{ar} \\ & \overline{bb} & \overline{bc} & \overline{br} \\ & & \overline{cc} & \overline{cr} \\ & & & \overline{rr} \end{array}$$

$\overline{rr}$ is only needed for the final control.

Control.

Let $a+b+c=S$

further in the same way as before

$$a_1S_1+a_2S_2+\ldots+a_nS_n=\overline{aS}, \text{etc.}$$

Then we must have

$$\begin{aligned} \overline{aa}+\overline{ab}+\overline{ac}&=\overline{aS} \\ \overline{ab}+\overline{bb}+\overline{bc}&=\overline{bS} \\ \overline{ac}+\overline{bc}+\overline{cc}&=\overline{cS} \\ \overline{ar}+\overline{br}+\overline{cr}&=\overline{rS} \end{aligned}$$

The normal equations for the determination of α, β, γ are then:

$$\begin{aligned} \overline{aa}.\alpha+\overline{ab}.\beta+\overline{ac}.\gamma&=\overline{ar} \\ \overline{ab}.\alpha+\overline{bb}.\beta+\overline{bc}.\gamma&=\overline{br} \\ \overline{ac}.\alpha+\overline{bc}.\beta+\overline{cc}.\gamma&=\overline{cr} \end{aligned}$$

The solution proceeds as follows:

The following calculations are made—

$$\overline{bb}-\frac{\overline{ab}}{\overline{aa}}\overline{ab}=\overline{bb_1}$$

$$\overline{bc}-\frac{\overline{ab}}{\overline{aa}}\overline{ac}=\overline{bc_1}$$

$$\overline{br}-\frac{\overline{ab}}{\overline{aa}}\overline{ar}=\overline{br_1}$$

$$\overline{cc}-\frac{\overline{ac}}{\overline{aa}}\overline{ac}=\overline{cc_1}$$

$$\overline{cr}-\frac{\overline{ac}}{\overline{aa}}\overline{ar}=\overline{cr_1}$$

Finally—

$$\overline{cc_1}-\frac{\overline{bc_1}}{\overline{bb_1}}\overline{bc_1}=\overline{cc_2}$$

$$\overline{cr_1}-\frac{\overline{bc_1}}{\overline{bb_1}}\overline{br_1}=\overline{cr_2}$$

Calculate out—

$$\overline{bS}-\frac{\overline{ab}}{\overline{aa}}\overline{aS}=\overline{bS_1}$$

$$\overline{cS}-\frac{\overline{ac}}{\overline{aa}}\overline{aS}=\overline{cS_1}$$

$$\overline{rS}-\frac{\overline{ar}}{\overline{aa}}\overline{aS}=\overline{rS_1}$$

Then the following must be true—

$$\begin{aligned} \overline{bb_1}+\overline{bc_1}&=\overline{bS_1} \\ \overline{bc_1}+\overline{cc_1}&=\overline{cS_1} \\ \overline{br_1}+\overline{cr_1}&=\overline{rS_1} \end{aligned}$$

$$\overline{cS_1}-\frac{\overline{bc_1}}{\overline{bb_1}}\overline{bS_1}=\overline{cS_2}$$

$$\overline{rS_1}-\frac{\overline{br_1}}{\overline{bb_1}}\overline{bS_1}=\overline{rS_2}$$

Then we must have—

$$\overline{cc_2}=\overline{cS_2} \quad \text{and} \quad \overline{cr_2}=\overline{rS_2}$$

From these the unknown quantities a, β, γ are obtained

$$\gamma=\frac{\overline{c\mathfrak{r}_2}}{\overline{cc_2}},\quad \beta=\frac{\overline{b\mathfrak{r}_1}}{\overline{bb_1}}-\gamma\frac{\overline{bc_1}}{\overline{bb_1}},\quad a=\frac{\overline{a\mathfrak{r}}}{\overline{aa}}-\beta\frac{\overline{ab}}{\overline{aa}}-\gamma\frac{\overline{ac}}{\overline{aa}}$$

Control of the whole Calculation.—Let the substitution of a, β, γ in the original equations leave the errors—

$$\mathfrak{f}_1=aa_1+\beta b_1+\gamma c_1-\mathfrak{r}_1,\qquad \mathfrak{f}_2=aa_2+\beta b_2+\gamma c_2-\mathfrak{r}_2,\ \text{etc.}$$

Then we must have

$$\mathfrak{f}_1^2+\mathfrak{f}_2^2+\ldots+\mathfrak{f}_n^2=\overline{\mathfrak{rr}}-\frac{\overline{a\mathfrak{r}}}{\overline{aa}}\overline{a\mathfrak{r}}-\frac{\overline{b\mathfrak{r}_1}}{\overline{bb_1}}\overline{b\mathfrak{r}_1}-\frac{\overline{c\mathfrak{r}_2}}{\overline{cc_2}}\overline{c\mathfrak{r}_2}$$

The weights of the values of a, β, γ thus obtained are

$$p_\gamma=\overline{cc_2},\quad p_\beta=p_\gamma\frac{\overline{bb_1}}{\overline{cc_1}},\quad p_a=p_\gamma\frac{\overline{ca}\,.\,\overline{bb_1}}{\overline{cc}\,.\,\overline{bb}-\overline{bc}\,.\,\overline{bc}}$$

The squares of the mean errors of a, β, γ are obtained by dividing the expression $\left(\mathfrak{f}_1^2+\mathfrak{f}_2^2+\ldots\mathfrak{f}_n^2\right)\div(n-3)$ by p_a, p_β, p_γ; 3 is here the number of the constants determined.

In case the corrections a, β, γ thus obtained by the use of least squares are not sufficiently accurate, the values A_0+a, $B_0+\beta$, $C_0+\gamma$, which result must be again considered as approximations to A, B, C and the calculation repeated.

Plan of the Calculation.—It is advisable when making these calculations to write the numbers always in the same order, *e.g.* (omitting the lines over the sums of the products).

$\log aa$		
$\log ab$	$\log bb_1$	
$\log ac$	$\log bc_1$	$\log cc_2$
$\log aS$	$\log bS_1$	
$\log a\mathfrak{r}$	$\log b\mathfrak{r}_1$	$\log c\mathfrak{r}_2$
$\log\frac{ab}{aa}$	$\log\frac{bc_1}{bb_1}$	$\log\frac{c\mathfrak{r}_2}{cc_2}$
$\log\frac{ac}{aa}$	$\log\frac{b\mathfrak{r}_1}{bb_1}$	
$\log\frac{a\mathfrak{r}}{aa}$		

aa	ab	ac	aS	$a\mathfrak{r}$		
	bb	bc	bS	$b\mathfrak{r}$		
	$\frac{ab}{aa}ab$	$\frac{ab}{aa}ac$	$\frac{ab}{aa}aS$	$\frac{ab}{aa}a\mathfrak{r}$		
Diff. =	bb_1	bc_1	bS_1	$b\mathfrak{r}_1$		
		cc	cS	$c\mathfrak{r}$	$\mathfrak{r}S$	$\mathfrak{rr}$
		$\frac{ac}{aa}ac$	$\frac{ac}{aa}cS$	$\frac{ac}{aa}a\mathfrak{r}$	$\frac{a\mathfrak{r}}{aa}aS$	$\frac{a\mathfrak{r}}{aa}a\mathfrak{r}$
	Diff. =	cc_1	cS_1	$c\mathfrak{r}_1$	$\mathfrak{r}S_1$	
		$\frac{bc_1}{bb_1}bc_1$	$\frac{bc_1}{bb_1}bS_1$	$\frac{bc_1}{bb_1}b\mathfrak{r}_1$	$\frac{b\mathfrak{r}_1}{bb_1}bS_1$	$\frac{b\mathfrak{r}_1}{bb_1}b\mathfrak{r}_1$
	Diff. =	cc_2	cS_2	$c\mathfrak{r}_2$	$\mathfrak{r}S_2$	
						$\frac{c\mathfrak{r}_2}{cc_2}c\mathfrak{r}_2$

4.—Corrections and the Calculation of Corrections.

The result sought is almost never given directly by the observations; much more frequently these are affected by circumstances which must not be neglected in accurate determinations. With greater pretensions to accuracy, the number of influencing circumstances which must be considered increases as well as the difficulty of eliminating them, so that frequently the most important part of the work is introduced by these *corrections.* Hence also it is necessary first to determine their influence, and then to take them into the calculation, so far as is necessary, in as simple a manner as possible. How far we can go in taking account of corrections depends of course upon the limit which is here imposed upon us by the deficiencies of the observations, as well as by our incomplete knowledge of the laws of nature and of the numerical values which they involve. But, on the other hand, it is frequently unnecessary to carry the accuracy of the correction to this limit. Plainly it is always sufficient to attain to such a degree of accuracy that the neglected part of the corrections is materially less than the possible influence of the errors of observation upon the result. Hence we may use for corrections abbreviated methods similar to those used for errors. Practice in these calculations is an essential condition for accurate and yet ready physical work.

One of the simplest physical measurements, for example, is weighing or determining the mass of a body. Here we have first the errors of observation, which are made up of those due to the inaccuracy of our readings and of our judgment about them, and to some faults of the balance which cannot be calculated—as friction, change of the ratio of the balance-arms, etc. It is also impossible to prepare a set of weights free from error, or even to draw up perfectly true correction tables for them. As we do not suppose specially good instruments or accurate observations, other errors unavoidable, but determinable in their amount, which therefore can be eliminated from the result, become noticeable. It is therefore always requisite to take account of them, where we make any pretensions to accuracy. To this part belongs first

the inequality of the arms of the balance, which, at least with large weights, has usually a marked influence. It is eliminated by the rules given in **11.**

But, secondly, the weights and the body weighed suffer a loss of weight on account of the air which they displace, which—even in the use of ordinary shop-scales which show 1 grm. with loads of 1 kgr.—may become greater than the errors of weighing. In order, now, to reduce to the weight *in vacuo,* we must know the density of the air, a magnitude which may vary within certain limits. But although the complete neglect of the correction is only admissible in a very rough weighing, it is, on the other hand, easily seen that for common use, even in scientific investigations, the alteration of the density of the air is not of sufficient importance to be considered, and we may give a mean value to the correction. If, therefore, we confine ourselves to a correspondingly approximate calculation of the correction, a very considerable improvement of the result may be effected in about a minute.

The labour is somewhat greater, if the mean value will not be sufficient. In this case, the temperature and height of the barometer, at least, must be observed. The observed height of the barometer must not, however, be taken as the real height, since both the mercury and the scale expand by heat; this expansion must be taken into consideration. The variation of gravity on the earth's surface would also have to be brought into the calculation. Finally the density of the air varies with the humidity, and therefore in very accurate weighings this also must be taken into account.

Now, if all these observations and calculations were carried out with complete accuracy, they would become very laborious. But after we have informed ourselves as to what degree of accuracy we desire or can attain in the result, and as to the influence of the corrections, we can determine what degree of approximation is admissible or necessary, and, with some practice, attain the result with small trouble.

In the same way, corrections come into most physical problems. It is especially the changing *temperature* which, in many ways, influences the measurements, and therefore frequently furnishes a reason for corrections.

It is usually possible to make use of the processes described on p. 10 and the formulæ for approximation there given, for shortening the calculation of corrections.

EXAMPLES.

(1.) It is well known that we call $3a$ the coefficient of the cubical expansion of a body, when a is used for the linear coefficient. Strictly speaking, when the linear dimensions are varied in the ratio $1 + at$, the volume changes in the ratio $(1 + at)^3 = 1 + 3at + 3a^2t^2 + a^3t^3$. But for almost all solid bodies $a < 0{\cdot}00003$, so that even for a change of temperature of 100°, the neglected part $3a^2t^2 < 0{\cdot}000027$, or $\frac{1}{37000}$ of the total. Therefore it is only when such small quantities are under consideration that the abbreviated calculation must not be employed. Then, however, it must also be taken into the calculation that the coefficient of expansion itself varies a little with the temperature. The term a^3t^3 is entirely without noticeable influence.

(2.) In **21** we treat the expansion of the mercury as a correction by putting $\frac{l}{1 + 0{\cdot}00018t} = l - 0{\cdot}00018lt$ (formula 4, p. 10), in the reduction of the height of the barometer to 0°. Here we neglect the higher powers of $0{\cdot}00018t$. But it will be seen that the next power amounts, for $t° = 30$, to only $0{\cdot}00003$; therefore multiplied by $l = 760$ mm., about $\frac{1}{45}$ mm., a quantity which may almost always be neglected.

On the other hand, it would frequently be inadmissible to treat the expansion of a gas, which is about twenty times greater, in a similar manner.

(3.) When the weight of a body has been determined by double weighing (**11**), and the weight has been found on the one side p_1, on the other p_2, the actual weight is, strictly speaking, $\sqrt{p_1 p_2}$. Instead of this geometrical mean, the arithmetical $\frac{1}{2}(p_1 + p_2)$ may without hesitation be used (formula 9, p. 11). For, calling $p_1 = p + \delta$, and $p_2 = p - \delta$, for which $p = \frac{1}{2}(p_1 + p_2)$, we have

$$\sqrt{p_1 p_2} = \sqrt{p^2 - \delta^2} = p\sqrt{1 - \frac{\delta^2}{p^2}} = p\left(1 - \frac{1}{2}\frac{\delta^2}{p^2}\right). \quad \text{(Formula 3.)}$$

Now the balance must be very badly adjusted for δ to be as much as $\frac{1}{1000}p$. In this case $\frac{1}{2}\frac{\delta^2}{p^2}$ would be half a millionth—a quantity which, in comparison with 1, need never be considered if such a balance be used.

Other examples will be found below in the different problems.

5.—INTERPOLATION FROM OBSERVATIONS.

A magnitude y, which depends on another x, often has to be determined for some exact value of x. Similarly the problem to be solved by observation may consist in determining under what circumstances a certain well-defined position of the object of observation is produced. It is nevertheless often troublesome, and sometimes indeed impossible, to regulate the circumstances to the quite accurate fulfilment of this condition. Thus it is usually difficult to maintain the temperature of a body accurately at a prescribed degree at which, perhaps, its volume, its elasticity, or its electrical conductivity, is to be determined; in a weighing, to employ such weights that the index stands accurately at zero, is tedious and in some circumstances unattainable. This is also the case when galvanic resistances have to be so balanced that a galvanometer needle points to a definite division, for instance to 0°.

In such very frequent cases it is often possible from neighbouring observations to interpolate the exact relation required, and thus to attain essential advantages in the simplicity of the required instruments, in the expenditure of time, and even in exactness.

Let x_0 be the point at which the instrument should stand, and y_0 the required quantity corresponding to x_0. Instead of y_0 we have actually observed

y_1 giving the result x_1
y_2 ,, ,, x_2

If these two positions lie so near to each other and to x_0 that within these limits the variation of x is proportionate to y, one has obviously

$$(y_0 - y_1) : (x_0 - x_1) = (y_2 - y_1) : (x_2 - x_1)$$

whence

$$y_0 = y_1 + (x_0 - x_1)\frac{y_2 - y_1}{x_2 - x_1}$$

It is most advantageous to take x_1, and x_2 at opposite sides of x_0.

For examples, among others, see **8** and **70**.

If the proportional increase of y and x which we have assumed is not fulfilled, the interpolation is less simple. If the law of the increase is not known, at least three observations near together are required. Formulæ for numerical interpolation are given by Lagrange and Gauss (see Weinstein, *Physikalische Massbestimmung*, § 291); but in this case a graphic method of interpolation is mostly used. The observed values of x and y are plotted, as abscissæ and ordinates, on paper ruled in squares, the points obtained (crosses form the best marks) are joined by a curve, and on this the value y_0 which corresponds to the abscissa x_0 is measured. Errors of observation are rendered visible, where many observations are made, as irregularities in the curve. This construction may be used to eliminate the errors, but must be employed cautiously.

6.—Rules for the Numerical Calculations.

The numerical calculation of the result can only be performed with a limited number of figures, a circumstance which renders absolute accuracy in the calculations impossible. This would also in most cases be of no advantage.

It is generally well to keep to the rule that the result is to be brought out to so many figures, that the last of them, on account of the errors of observations, makes no pretension to accuracy, but that the last but one may be taken as pretty accurate. In doubtful cases one place too many should be taken rather than one too few.

All the figures, however, should be correct as to the working. Hence it follows that a long calculation, *e.g.* with logarithms, must be gone through with at least one place more than is to be given in the result, for by the neglecting of further figures the last place may by degrees become wrong to the amount of several units. The extra place is discarded in the final result, increasing the last but one by 1 if the figure rejected be 5 or more.

Of course ciphers added, or those prefixed to a decimal, will not be counted in the number of figures.

Mistakes are made in this respect, especially by beginners, in the most different directions.

For example, let the volume v of a rectangular body be determined by measurements of its three dimensions. Let these be 10·5, 15·7, 30·9. The accurate numerical result would be $v = 5093·865$ cubic mm. It would be labour thrown away to calculate at such length, and useless to record such a result. For, suppose an error of $\frac{1}{20}$ mm. had been made in the measuring, the result might have been 50 cubic mm. too great or too small! It suffices, therefore, to reckon $v = 5090$, or at most 5094, and to abbreviate the multiplication, or to use 4 figure logarithms.

On the other hand, divisions are often carried out to too *few* places. The same observer who, as above, multiplied to 7 figures, determines, say, a specific gravity by weighings to $\frac{1}{10}$ mgr. with a delicate balance, and contents himself with calculating it as 2·5, whereas he might probably have been correct to the 4th decimal place. The beginner in physical measurement should carefully see to the right handling of the figures, and for this purpose take note of the rules in **2** and **4**.

WEIGHT AND DENSITY.

7.—Adjusting and Testing a Balance.

By the weighing of a body is meant the determination of the ratio of its weight or its mass to a unit of weight or mass, *e.g.* the gram.

We assume that we have a correct set of weights, which permit the divisions and multiples of the unit to be made up. On the reduction of the weighing to vacuo, and the correction of a set of weights, see **11** and **12**.

The descriptions which follow refer, so far as any special construction is kept in view, to the form of the balance commonly used in chemical analysis.

Adjustment of the Balance.—There is usually a level or plumb-line attached to the balance-stand by the maker, which is adjusted by means of the foot-screws. Where this arrangement is wanting, a level is placed in the balance case, and the adjustment attained by it.

The beam is now released, and, correcting any excess of weight on either side, the observer makes sure that the balance has a stable position of equilibrium. Should the equilibrium be unstable (the balance "*set*"), the movable weight in the middle must be screwed down until the defect is remedied.

The *sensitiveness* can be regulated by screwing up the movable weight as far as is necessary. The time of an oscillation increases with the increase of sensitiveness; it should be made, in balances of the ordinary form, about 10 to 15 seconds (in Bunge's and other short-beamed balances 6 to 10

seconds). Slower oscillations occasion loss of time in weighing, and usually cause irregularities in the adjustment which make the larger deflection useless.

When a suitable time of oscillation has been attained, the pointer is made to point to the middle division of the divided scale, or swing equally on both sides of it, when the balance is unloaded. This is effected by the arrangement provided for the purpose (screws or movable weights at the end of the beam, slot in the central screw-head, or movable arm). When the desired object is nearly attained by means of the movable weight, the last fine adjustment to the zero may be made with the foot-screws, shortening one by as nearly as possible the same amount that we lengthen the other.

Testing the Balance.—The balance must fulfil the following conditions before it is taken into use:—

i. It must, when repeatedly stopped and again released, always take up the same position (it must be seen that the three knife-edges are carefully cleaned).

ii. When the balance is swinging freely, the distance it swings should diminish but slowly, and this condition must be fulfilled even with the greatest admissible load.

iii. When the stopping apparatus is raised, the pointer should stand immediately over the middle division; and when it is lowered, the two pins on which the beam rests when stopped should release it at the same time.

iv. The equality of the arms should then be made sure of, which is known by the fact that weights (which should not be too small), which are in equilibrium, produce the same position in the balance when they are interchanged with each other (**10**).

v. The effect of a weight must be unaltered, in whatever part of the scale-pan it is placed. In table balances and those of similar construction great errors may be introduced from this cause. Even in the ordinary balances slight differences occur which are most simply avoided by the introduction of a loose link between the scale-pan and its support.

The following minor points are to be considered in procuring a balance:—The apparatus for moving the rider should

be provided with stops to prevent it striking the beam. It should also, as well as the apparatus for stopping the beam, and the doors of the case, have an easy quiet motion. To avoid parallax in reading, the extremity of the pointer should move very closely in front of the divisions, or, still better, above them. As to the size of the divisions, about a millimeter is to be recommended. It is less important that the two pans should be of the same weight, than that the shorter pan provided for specific gravities should be accurately equal in weight to one of the longer ones.

Use of the Balance.—It should stand on a table protected from the tremors of the floor. If it is impossible to help weighing in a heated room, or one into which the sun shines, we must at least protect the balance from unequal heating. To preserve from rust, and to exclude as much as possible the influence of hygroscopic moisture during weighing, a vessel filled with caustic potash or calcium chloride is placed in the balance-case.

Weights must only be put on when the balance is stopped. When the larger weights are to be put on, or when the load is to be taken off the balance, the pans also should be stopped if any apparatus is provided for the purpose. Swinging of the scales during weighing may give rise to errors. After every weighing with large weights we must make sure that the zero-point is unaltered, or make a new determination of it. It is a matter of course that the final weighings are performed with the case shut.

For the most delicate weighings a mirror attached to the balance beam may be used instead of the pointer, the position being read with a telescope on a scale. (See **48** to **50**.)

8.—Weighing by observing the Swinging of a Balance.

The ordinary operation of weighing in which weights are added, and at last the rider is moved until the pointer of the balance swings equally on both sides of the middle division, has several defects. Firstly it demands, on account of the unavoidable alteration of the zero point, frequent readjustment

of the balance, which takes up much time. In the next place, it is only applicable to balances provided with riders. Thirdly, the process takes a long time, and requires several careful observations, which nevertheless are of no use in determining the result. Finally, it is as a rule better to make a measure depend, not on a trial whether two quantities are equal, for equality is only approximately attainable, but on a trial as to how much they differ. The following method of weighing, by observing the oscillation of the pointer and by interpolation, escapes these objections.

The first thing to be done is the determination of the zero point, by which we mean the point of the scale at which the index comes to rest when the balance is unloaded. Since we cannot and should not wait until the motion ceases to determine this point directly, we must deduce it from observing the division to which the index attains when swinging. The extent of the swing may amount to from 2 to 5 scale divisions.

Where we require only moderate accuracy, it is enough to observe two successive points at which the index turns, and to take their arithmetical mean. It is better to observe several points of turning on both sides, taking care, for the sake of simplifying the reductions, that the first and last shall be on the same side, *i.e.* we make an uneven number of observations. Five or seven are always enough. We then take the arithmetical mean of the observations on the one side, that is, the 1st, 3rd, 5th; and of those on the other, viz. the 2nd and 4th; and again take the mean of these two numbers. This is the required zero-point. In order not to have to distinguish the deviations to the right and left by signs, we call the middle point of the scale not 0 but 10.

Example—

	Turning-points.					Means.	Zero.
No.	1.	2.	3.	4.	5.		
	10·4		10·3		10·3	10·33	9·74
		9·1		9·2		9·15	

We now place the body on the one scale, and bring the balance nearly to the zero-point (within one or two scale

divisions) by weights in the other, and at last by moving the rider from one division of the beam to another, and make another set of observations of the swinging as above, then take off or add one or more milligrams, according as the weights were too heavy or too light, until the position of equilibrium falls on the other side of the zero-point, and determine it by again observing the excursions of the index.

The required weight p_0 of the body—*i.e.* the number of weights that must be added in order that the balance when loaded may settle to the zero-point—is given, by a simple interpolation from these observations (**5**).

Let there have been found

the zero	e_0
with the weight P the position	E
with the weight p the position	e

we have, since for small deflections the difference of the positions of equilibrium is proportional to the difference of the weights—

$$\frac{e_0 - e}{E - e} = \frac{p_0 - p}{P - p}$$

therefore

$$p_0 = p + (P - p)\frac{e_0 - e}{E - e}$$

The above differences must all be taken with the proper sign, on which account it is simpler to have the scale divisions so numbered that increased reading corresponds to increased weight.

The operation may also be expressed somewhat more simply, thus:—The two observations with different weights give the difference $d = (E - e)/(P - p)$ of position (the deviation), which corresponds to 1 mgr. increase of the weight (*the sensitiveness*). If, further, we determine by subtraction the number of divisions $A = (e_0 - e)$ at which the point of equilibrium is from the zero-point with one of the weights (it is immaterial which, but to simplify the calculation the nearest to the zero is usually chosen), the number of milligrams which must be added (or subtracted), so that the balance may settle to the zero, is given by division $= A/d$. Compare also the beginning of the next article.

Example.—The value for the zero has been determined above to be 9·74.

Weight. mgr.	Turning-point.			Mean.	Point of Rest.
3036	7·8	7·8	7·9	7·83	9·04
	10·3	10·2		10·25	
3037	9·5	9·4	9·3	9·40	9·95
	10·5	10·5		10·50	

Deviation for 1 mgr. = 0·91 scale division.

3037 mgr. were accordingly too heavy by

$$\frac{9{\cdot}95 - 9{\cdot}74}{0{\cdot}91} = \frac{0{\cdot}21}{0{\cdot}91} = 0{\cdot}23 \text{ mgr.}$$

$$\therefore \quad \text{weight } p_0 = 3036{\cdot}77 \text{ mgr.}$$

Or by the previous formula,

$$p_0 = 3036 + \frac{1 \times 0{\cdot}7}{0{\cdot}91} = 3036{\cdot}77.$$

With a little practice time is saved by this method of observation, since the performance of the reductions soon becomes quite mechanical, whilst the accuracy is greater than in the ordinary method.

It is immaterial whether the weights are reckoned in grams or milligrams, but one settled method should be kept to. The recording of the observations should also be kept in a regular form as above.

9.—Determination of the Sensitiveness of a Balance.

By the sensitiveness of a balance we mean the difference of indication for 1 mgr. difference in the weight. The determination of this quantity with various loads is important as a criterion of the excellence of the balance, and further, as a means of simplifying the process of weighing. For if we possess a table in which the change of position for 1 mgr., when different weights are on the balance, is given, it is enough for each weighing, in addition to determining the zero-point, to make one single observation of the position with nearly the right weight.

The method of procedure is self-evident. The load for which the sensitiveness is to be determined is put into each

pan, and into one a small excess, so that the position of equilibrium is from 2 to 4 divisions from the centre. This position is accurately determined according to **8**; we call it e. Now, by adding a weight of a milligrams to the other pan, the position of equilibrium is to be brought to about as far on the other side of the centre, and observed as before. If this position be called e', the required sensitiveness is

$$\frac{e - e'}{a}$$

When this quantity has been determined for different loads (with the ordinary balances used in analysis, at intervals of 10 grms.), the results are entered graphically on paper ruled in squares, the loads as abscissæ, the sensitiveness as ordinates. Through the resulting points draw a curve, from which the sensitiveness may be determined for any particular load.

On the regulation of the sensitiveness see **7**.

The dependence of the sensitiveness on the load arises from the relative positions of the middle and end knife-edges. On the grounds of convenience a sensitiveness independent of the load is to be desired in fine balances, which requires the three edges to be in the same plane. But as this condition can, strictly speaking, be fulfilled for only one definite weight, on account of the bending of the beam, the best makers are accustomed to produce it for a mean load. Hence there is at first a slight increase of sensitiveness with increased load, and then for still greater weights a decrease.

10.—Determination of the Ratio of the Arms of the Balance.

The two arms of the balance are inversely as the weights, which, when placed in the corresponding pans, bring the balance to the zero-point (**8**). Since, usually, the absolute accuracy of the set of weights cannot be assumed, the following method is used:—

The zero is observed; then in each pan weights are placed of the same nominal value, about equal to half the maximum which the balance will carry, and made equal by adding

milligram weights, or moving the rider until the balance is in equilibrium, in which proceeding we should, for the sake of accuracy, use the method of interpolation (**8**). The zero-point must be sufficiently often tested, and if any small alteration is found, the mean of the positions before and after the weighing should be used. Then the weights are interchanged and again made equal. If we call the two nominally equal weights p and P, and have found that the balance is in equilibrium when

	Left. Right.
in the first weighing	$p + l = P$
in the second weighing	$P = p + r$

we have, if L and R denote the lengths of the arms of the balance left and right—

$$\frac{R}{L} = 1 + \frac{l - r}{2p}$$

A small excess of weight on one pan may be considered as a negative weight on the other (see Example).

The ratio of the arms can also be deduced from the double weighing of a body (see **11**, 1).

Proof.—According to the law of the lever—

$$L(p + l) = R.P.$$
$$L.P = R\,(p + r),$$

from which, by formulæ 8 and 3, p. 10, we have

$$\frac{R}{L} = \sqrt{\frac{p + l}{p + r}} = \sqrt{\frac{1 + l/p}{1 + r/p}} = 1 + \frac{l - r}{2p}$$

Example.—

Left.	Right.
(50 grms.)	$(20 + 10 + \ldots) + 0{\cdot}83$ mgr.
$(20 + 10 + \ldots)$	$(50) + 2{\cdot}56$ mgr.

$$l = -0{\cdot}83 \qquad r = 2{\cdot}56$$

$$\frac{R}{L} = 1 + \frac{-0{\cdot}83 - 2{\cdot}56}{100000} = 1 - 0{\cdot}0000339$$

or

$$\frac{L}{R} = 1{\cdot}0000339.$$

In the above example the figures in brackets signify the weights marked with those figures.

From the determination it follows immediately (see **11**)—

$$(50) = (20 + 10 + \ldots) - 0{\cdot}86 \text{ mgr.}$$

11.—ABSOLUTE WEIGHING OF A BODY.

I. ELIMINATION OF THE INEQUALITY OF THE ARMS OF THE BALANCE.

The apparent weight as found by the weighing is multiplied by the ratio of the arms of the balance, using as numerator the length of that arm to which the weights are attached. The following methods make us independent of this ratio, which, moreover, in accurate weighings, cannot be considered invariable.

1. *Double Weighing.*—The body is weighed first on the right-hand scale, then on the left-hand one. If p_1 and p_2 be the weights which must be placed on the right-hand and left-hand pans respectively to balance the body, the required weight P of the body is the arithmetical mean $P = \frac{1}{2}(p_1 + p_2)$. The zero-point of the balance need not be determined.

From this we can also immediately find the ratio of the arms of the balance if p_1 and p_2 are referred to the true zero of the balance (Formula 3, p. 10).

$$\frac{\mathrm{R}}{\mathrm{L}} = \sqrt{\frac{p_2}{p_1}} = \sqrt{1 + \frac{p_2 - p_1}{p_1}} = 1 + \frac{p_2 - p_1}{2p_1}$$

2. *By Taring.*—The body being upon one pan is balanced by loading the other pan in any convenient way; it is then taken away, and weights are put in its place until the former reading of the balance is obtained. The weights put on give the weight of the body.

II. REDUCTION TO THE WEIGHT IN VACUO.

The object of weighing a body is to determine its mass, *i.e.* its equality with the mass of the weights. Instead of equality of *masses* we may also say equality of *weights* provided that the weighing is performed *in vacuo*, in which case the mass

of a body is proportional to its weight. In the air both the body and the weights lose weight equal to the weight of the air which they respectively displace.

If we call

m = the apparent weight of the body in air, *i.e.* the weights which balance it in the air;
λ = the density of the air. ($\lambda = 0{\cdot}0012$ as a mean value. See also **15** and Table 6);
s = the density (specific gravity) of the body;
δ = the density of the weights;

the weight *in vacuo* will be

$$M = m\left(1 + \frac{\lambda}{s} - \frac{\lambda}{\delta}\right)$$

There is therefore to be added to the apparent weight m a correction $m\lambda\left(\frac{1}{s} - \frac{1}{\delta}\right)$, which is so much the greater as the difference between s and δ is greater. It is almost always sufficient to use the mean value $0{\cdot}0012$ for λ. In this case the correction for brass weights may be taken from Table 8.

Proof.—The volume of the body is $V = M/s$, that of the weights $v = m/\delta$. Every body loses, in the air, the weight of the air which it displaces; the body therefore which we have weighed loses λV, the weights λv. Since the weights, after subtracting these losses, are equal, we have

$$M - \lambda V = m - \lambda v, \text{ or } M\left(1 - \frac{\lambda}{s}\right) = m\left(1 - \frac{\lambda}{\delta}\right)$$

therefore, on account of the smallness of λ in comparison with s or δ, we have, by formula 8 (p. 11)

$$M = m\frac{1 - \frac{\lambda}{\delta}}{1 - \frac{\lambda}{s}} = m\left(1 + \frac{\lambda}{s} - \frac{\lambda}{\delta}\right)$$

Example.—The correction of the apparent weight w of a quantity of water when weighed with brass weights ($\delta = 8{\cdot}4$), amounts to $w \,.\, 0{\cdot}0012\left(\frac{1}{1} - \frac{1}{8{\cdot}4}\right) = w \,.\, 0{\cdot}00106$, *i.e.* 1·06 mgr. in every gram.

Where the question is not of the absolute weights, but only of ratios of weights, as in chemical analysis, the loss of weight of the substance in the air must still be taken account of, though that of the weights may be neglected.

If, for example, a dilute solution of silver be analysed by weighing a quantity of the solution and the silver chloride (density $= 5 \cdot 5$) obtained from it; and if P and p be the observed weights, these are, reduced to vacuo, $P(1 + 0 \cdot 0012)$ and $p\left(1 + \frac{0 \cdot 0012}{5 \cdot 5}\right)$. The proportion of silver chloride amounts therefore to

$$\frac{p \,.\, \left(1 + \frac{0 \cdot 0012}{5 \cdot 5}\right)}{P \,.\, (1 + 0 \cdot 0012)} = \frac{p}{P}\left[1 - 0 \cdot 0012(1 - \frac{1}{5 \cdot 5})\right] = \frac{p}{P} \times 0 \cdot 999.$$

The uncorrected value p/P would therefore be about 0·1 per cent too great. The customary neglect of such a simple correction must, in view of the costliness of the balance, the care spent on the weighings, and the great pretension to accuracy implied in the large number of decimals generally given in the result, be considered inadmissible.

As to the question of principle whether the gram is a unit of weight or of mass, consult the introduction to the appendix on the absolute system of measurements. In ordinary cases of measurement it makes, as a rule, no difference whether we speak of weights or masses, no errors arise from either designation. For a chemical analysis, or any other operation giving a percentage composition, it is plainly quite immaterial whether we use weights or masses. Similarly we shall obtain the same numbers whether we speak of the specific gravity of a body or, under the name density, of its specific mass; provided that we compare this property of the body with that of water as unit, as is always the case. It is only when the weights serve actually for measuring Force, as in measuring Work, Pressure, Elasticity, that we must strictly distinguish the cases.

12.—Table of Corrections for a Set of Weights.

The operation of determining the errors of a set of weights usually depends on the performance of as many weighings as there are weights to be corrected, and on the formation of the same number of equations from them, from which the ratio of the arms of the balance, and that of the weights to each other, is deduced.

In the sets of weights commonly used in analysis, the manner of proceeding is as follows:—

The larger weights are distinguished as

$$50' \quad 20' \quad 10' \quad 10'' \quad 5' \quad 2' \quad 1' \quad 1'' \quad 1'''$$

A double weighing is performed with $50'$ on one side, and the rest of the weights on the other. Suppose it has been found that the balance is in equilibrium, *i.e.* the pointer is in the same position as when the balance is unloaded, when

Left.	Right.
$50'$	$20' + 10' + \ldots + r$ mgr.
$20' + 10' + \ldots + l$ mgr.	$50'$,

then the ratio of the arms of the balance is **(10)**

$$\frac{R}{L} = 1 + \frac{l - r}{100{,}000}$$

and $$50' = 20' + 10' + \ldots + \tfrac{1}{2}(r + l)$$

In the same way $20'$ is compared with $10' + 10''$, and $10'$ with $10''$ and also with $5' + 2' + \ldots$ The ratio of the balance arms is somewhat dependent on the load, but when R/L has been determined, a single weighing is sufficient for the smaller weights. A weight p, on the right pan, is, on account of the length of the arms, reduced to pR/L when weighed on the left hand.

Example.—Let $r = -0{\cdot}63$ mgr. $\quad l = +2{\cdot}73$ mgr., then

$$50' = 20' + 10' + 10'' + 5' + 2' + 1' + 1'' + 1''' + 1{\cdot}05 \text{ mgr.}$$

$$\text{and } \frac{R}{L} = 1{\cdot}0000336.$$

Further, in the comparison of the 5 grm. weight with the sum of the small weights let it be found that the balance is in equilibrium when

Left.	Right.
$5' + 0{\cdot}31$ mgr.	$2' + 1' + 1'' + 1'''$

then on a balance with equal arms equilibrium would be obtained when the weights were

$$5' + 0{\cdot}31 \text{ mgr. and } (2' + 1' + 1'' + 1''')\,(1{\cdot}0000336)$$

$$\text{or } 2' + 1' + 1'' + 1''' + 0{\cdot}17 \text{ mgr.}$$

consequently

$$5' = 2' + 1' + 1'' + 1''' - 0{\cdot}14 \text{ mgr.}$$

Suppose that from these weighings we have found

$$\begin{aligned} 50' &= 20' + 10' + \ldots + A \\ 20' &= 10' + 10'' \qquad + B \\ 10'' &= 10' \qquad\qquad + C \\ 5' + 2' + 1' + 1'' + 1''' &= 10' \qquad\qquad + D \end{aligned}$$

where of course A, B, C, D may be either positive or negative.

From these equations the values of the 5 weights must be expressed in terms of some unit—the sum of the single grams being provisionally considered as one weight. If a comparison with a normal weight be not made at the same time, this unit is so chosen that the correction of the separate weights shall be as small as possible, which is the case when the whole sum is assumed to be correct—*i.e.* when we consider

$$50' + 20' + 10' + \ldots = 100 \text{ grm.}$$

Calling now $\frac{1}{10}(A + 2B + 4C + 2D) = S$

we have, as can be easily proved

$$\begin{aligned} 10' &= 10 \text{ grm.} - S \\ 10'' &= 10 \quad ,, \quad - S + C \\ 5' + 2' + \ldots &= 10 \quad ,, \quad - S + D \\ 20' &= 20 \quad ,, \quad - 2S + B + C \\ 50' &= 50 \quad ,, \quad - 5S + A + B + 2C + D = 50 + \tfrac{1}{2}A. \end{aligned}$$

The proof of the correctness of the numerical work is easily found from the above to be that the sum of the corrections, when expressed as numbers, must equal 0, and the four equations given by observation must be fulfilled.

Again, the following equations having been obtained by comparing the weights $5'$, $2'$, $1'$, $1''$, $1'''$ with each other.

$$\begin{aligned} 5' &= 2' + 1' + 1'' + 1''' + a \\ 2' &= 1' + 1'' \qquad\quad + b \\ 1'' &= 1' \qquad\qquad\quad + c \\ 1''' &= 1' \qquad\qquad\quad + d \end{aligned}$$

As in the previous case calling

$$\frac{1}{10}(a + 2b + 4c + 2d + S - D) = s$$

we have

$$\begin{aligned} 1' &= 1 \text{ grm.} - s \\ 1'' &= 1 \quad ,, \quad - s + c \\ 1''' &= 1 \quad ,, \quad - s + d \\ 2' &= 2 \quad ,, \quad - 2s + b + c \\ 5' &= 5 \quad ,, \quad - 5s + a + b + 2c + d. \end{aligned}$$

In the same manner we proceed with the smaller weights, only remarking that usually the inequality of the arms of the balance no longer needs consideration.

We have hitherto assumed the sums of the larger weights to be correct, in order to have corrections as small as possible. For most purposes (chemical analysis, specific gravity) which only require *relative* weighings, this assumption may be made. In order to refer the table of errors to an accurate gram weight, it is necessary to compare the weights, or one of them, with a normal weight (**11**). The calculation is easily got from the above.

A similar method of testing a series of weights of any other arrangement will be easily found.

To distinguish the weights of the same nominal value, the figures should be differently engraved, or they should be provided with distinguishing marks, otherwise accidental marks must be looked for. In the case of weights which consist of pieces of foil, the turning up of different corners may be made use of. No regard need be paid to the loss of weight from weighing in the air, for the larger weights are all of the same material, and with the smaller ones the difference is without noticeable influence. For testing the smaller pieces a lighter balance is, when possible, made use of, *i.e.* one which is more sensitive, with the same time of oscillation. The weighings are made by observations of the swing after (**8**), and the observation of the zero-point should be frequently repeated. It is customary to use the weights in a fixed order, so that each total weight will always be made up of the same individual weights; it is easy therefore to calculate the table of errors for the total weights by taking it for every 10000, 1000, 100, 10, and eventually for single milligrams.

13.—Density or Specific Gravity.

By the density or specific gravity of a solid or liquid (Tables 1 and 2) is meant the ratio of its mass to that of an equal volume of water at 4°. This latter has therefore the density 1, and the choice of any other temperature than 4° (such as 0° or 15°) must be pronounced unscientific, since the density of water at this temperature forms the basis of the metrical system.

Instead of the ratio of the masses we may use that of the weights *in vacuo*. If the meter and gram system of weights and measures be used, we may call the density the ratio of the weight to the volume, or, in the case of homogeneous bodies, the weight of unit-volume. In this case, mgr. and mm., gr. and cm., kgr. and dm., naturally belong to each other. Though the two terms density and specific gravity are in ordinary use synonymous, it should not be forgotten that in principle they are not identical.

The reciprocal of the density, *i.e.* the volume of unit mass, is called the specific volume of the substance. Atomic or molecular volume is the specific volume multiplied by the atomic or molecular weight of the body, or, what comes to the same thing, the atomic or molecular weight divided by the density.

A gas is measured, unless otherwise stated, at 0° and 760 mm. pressure. Mostly, however, a gas is compared, not with water, but with dry atmospheric air (for chemical purposes with hydrogen) of the same temperature and pressure, in which case the statement of the conditions is not needful. According to Avogadro's law equal volumes of different gases contain at equal pressure and temperature an equal number of molecules; in other words, the molecular volumes of all gases and vapours are equal (**16**).

I. Methods of Determining Densities.

(For corrections see under II. and III.)

A. *For Fluids.*

1. By weighing a volume measured in a graduated vessel such as a flask, tube, or pipette. The density of a quantity m grms., which has the volume v c.c., is $s = m/v$. On account of capillary attraction it is advisable to measure the volume in a graduated tube by difference, always reading off the position of the horizontal part of the surface (*cf.* **19**).

2. The weight m of the fluid filling a given vessel is determined, and also the weight w of the water filling the same vessel. Then the density $s = m/v$.

Any small flask filled to the brim or to a mark on the

neck easily affords results correct to the 3rd decimal. The vessels called Pyknometers, or specific gravity bottles, which are filled either completely or to a mark give better results. The third and fourth forms figured are the most accurate, in which one opening serves to admit the fluid, the other to let out (or suck out) the air. The 4th (Sprengel-Ostwald) is hung from the balance by a wire. The temperature is determined by immersion of the whole in a bath.

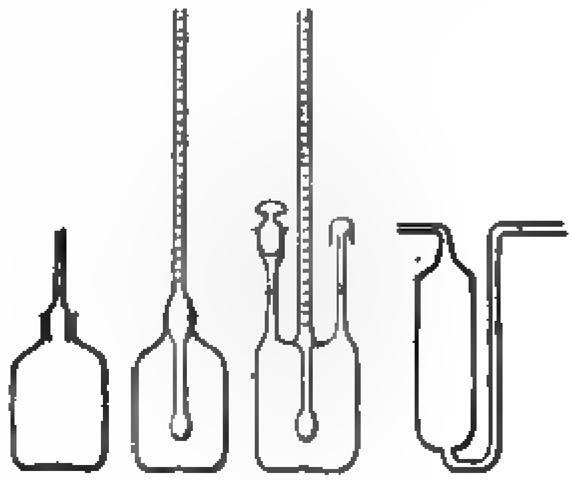


Fig. 1.

3. A body (*e.g.* a piece of glass) is weighed in the air, in the liquid, and in water. If we find the loss of weight in the liquid to be m, that in water w, we have again $s = m/v$.

Mohr's Balance.—A weight of glass is balanced on a beam divided decimally. The loss of weight of the body in water is equal to the weight of the largest rider; the others are 10, 100, and 1000 times lighter. The divisions of the beam on which the riders must be placed, in order to compensate for the loss of weight of the glass in any fluid, gives immediately the decimals of the specific gravity. The following conditions are necessary to its accuracy:—

i. The weights or riders must be in the proportion 1 : 10 : 100.

ii. The divisions must be at equal intervals; in order to test this a small balanced pan is hung on the other end of the beam, the heaviest rider placed on divisions 1, 2, etc., successively and the weights noted, which must be placed in the pan in order to balance. These also should be in the proportion 1 : 2, etc.

iii. The balance must, in water of temperature t, show that density which corresponds to t in Table 4. If instead of Q it reads Q_1 all results must be multiplied by Q/Q_1. A good Mohr's balance will give the fourth place with considerable accuracy.

4. Scale areometers (hydrometers) give by the division to which they sink either the density, or the specific volume, *i.e.*

the reciprocal of the density, the percentage of a solution, or the "so-called degrees of density." For the relation of these scales see Table 2. Areometers should be read through the fluid, the eye being level with the surface, so that it appears as a line. They should stand, in water of temperature t, at the number which corresponds to t on Table 4. Other points of the scale must be proved with fluids of known specific gravity.

5. The heights of columns of different fluids in tubes communicating with each other are, when equilibrium is established, in the inverse ratio of the densities.

B. *For Solids.*

1. WEIGHING AND MEASURING THE VOLUME.—If m grms. of the body measure v c.c. the density is $s = m/v$. The measuring, when the body is of a regular shape, may be done by a scale; when the body is irregular the volume may be measured by observing how much the surface of a quantity of liquid contained in a graduated tube rises when the body is put into it. This method is specially applicable to substances in small pieces. For substances soluble in water, some other fluid, *e.g.* alcohol, petroleum, or a saturated solution of the substance, may be used.

The volume may also be determined by immersing the body in water exactly filling a vessel with well-defined outflow and weighing the water which runs out.

2. PYKNOMETER.—Let the flask weigh P when filled with water, P' when the body is placed in it and the rest filled with water, while the body itself weighs m.

Then $w = P + m - P'$ and $s = m/w$.

The method is specially applicable in the case of small bodies, but then a flask as small as possible should be used.

3. WEIGHING UNDER WATER.—If m be the weight of the body, and it loses the weight w when weighed in water $s = m/w$.

With the Balance.—The body is hung to one of the pans of the balance by a thread or wire as thin as possible. The loss of weight of the wire is to be subtracted from w. This can easily be obtained by calculating the weight of the immersed part of the wire from the ratio of the immersed part to the whole length, and dividing by the density of the wire (Table 1).

When weighing in water the oscillations quickly decrease; the position is observed when the balance has come to rest. The thread should be as thin as possible, and only cut the surface of the water in one place, in order not to increase the capillary attraction, which otherwise would impair the accuracy of the weighing.

Bodies soluble in water are weighed in some other fluid of known density, and the result obtained as above multiplied by this density.

Bodies lighter than water are sunk by being fixed to another of sufficient weight, *e.g.* a metal clamp or bell of wire net under which the body is placed. The extra weight may remain in the water through all the weighings.

Instead of hanging the body to the pan of the balance, a vessel of water may be placed upon it, and the increase of weight determined when the body is suspended in it by a thread from a fixed stand. This increase is equal to the apparent loss of weight of the body in water.

With Nicholson's Hydrometer.—The upper pan is weighted at each operation until the instrument sinks to a mark on the stem—(1) by weights, (2) by the body + weights, (3) by weights while the body is under water lying on the lower pan; (1) minus (2) gives the weight of the body, (3) minus (2) the weight of the water displaced. Changes of temperature impair the accuracy the more the smaller the body is as compared with the hydrometer. Rubbing the stem with spirits of wine makes the certainty of the adjustment greater.

With Jolly's Spring Balance.—A spiral wire carries two pans hung one above the other, one always immersed in a vessel of water. We observe, exactly as with the hydrometer, the weights which must be put upon the upper pan to bring a mark upon the lower end of the wire to the same position. As fixed index, a mark upon a piece of looking-glass may be used to avoid parallax. It is possible to read to $\frac{1}{10}$ mm.

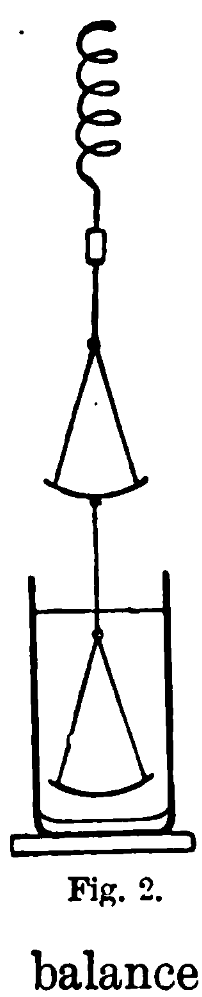


Fig. 2.

A simpler method of weighing with the spring balance without weights depends on the fact that the elongation x is

nearly proportional to the weight p by which it is produced, so that $p = A.x$. We can determine A once for all by means of a known weight. Since in determinations of density the unit of weight is eliminated we can take the scale division of the balance as unit. If the balance sinks by the laying on of the body on the upper pan, x divisions and x', when it is on the pan under water we have therefore $s = x/(x - x')$.

More accurately $p = Ax + Bx^2$. A and B are determined by two experiments, in one of which the load produces about the maximum depression, the other about half. From these a table can be easily constructed which gives the load corresponding to any depression.

In all cases air-bubbles adhering to the bodies must be removed by repeatedly dipping them in and taking out again or by the application of a brush.

II. Density. Reduction of the Weight to that of Water at 4° and in vacuo.

The methods for determining densities given in A and B under 2 and 3, require a correction, which is applied according to the following general rule:—

We must first take account of the fact that the water is usually at some other temperature than $+4°$. The density Q is found from the temperature by the help of Table 4. In the second place, the weighings are to be reduced to weighings *in vacuo*. Table 6 gives the density λ of dry air for various temperatures and pressures. It is almost always enough to take the mean value $\lambda = 0{\cdot}0012$. The neglect of the expansion of the water may affect the result to the extent of $\frac{1}{3}$ per cent, the loss of weight in air about 2 in the second place of decimals.

We will call

Q the density of the water employed (Table 4);
λ the density of the air compared with water (mean value $\lambda = 0{\cdot}0012$);
m the apparent, *i.e.* uncorrected, weight of the body in air, or, in the case of a fluid, the apparent loss of weight of a body immersed in it;
w the apparent weight of the volume of water of density Q, equal to the volume of the body.

As w we may therefore have

1. In the case of fluids—the observed weight of the water in specific gravity bottle, or of that displaced by the piece of glass weighed in the fluid and water.

2. In the case of solids—the observed loss of weight of the body in water when a determination is made by the method of Archimedes with balance or areometer; or the weight of the water displaced by the body when the specific gravity bottle is used.

m/w is the rough, uncorrected specific gravity; the accurate value is

$$s = \frac{m}{w}(Q - \lambda) + \lambda, \quad \text{or} = \frac{m}{w}Q + \left(1 - \frac{m}{w}\right)\lambda$$

For the calculation see also the next section.

It will be seen that the influence of the loss of weight in the air vanishes when the density is 1. It reaches in the case of platinum ($m/w = 21$) the value 0·024. If, in addition, the expansion of the water by temperature be neglected, the result may be too great by about 0·08.

(See R. Kohlrausch, *Praktische Regeln zur genaueren Bestimmung des specifischen Gewichtes*, Marburg, 1856.)

Proof.—If the body, solid or fluid, have the weight m in the air, and displace the quantity of air l, it would, *in vacuo*, weigh $m + l$. In considering the determination of w we may distinguish three cases. If we have determined the weight of an equal volume of water by weighing, the weight *in vacuo* is $w + l$. If we have measured the apparent loss of weight w of a solid by immersion in water, this must be increased by l, since the weight *in vacuo* would be so much greater than in the air. In the same way, thirdly, if we have determined the density of a fluid by finding the apparent loss of weight of the same body when weighed successively in water and the fluid, each of these must be increased by l.

If, however, the water have not the density 1, but Q, the same volume of water at 4° would weigh $(w + l)/Q$. In all cases, therefore, the true density s of the body is obtained by the formula—

$$s = \frac{m + l}{w + l}Q$$

But since $(w + l)/Q$ is the volume of the displaced air, calling its density (compared with water) $= \lambda$

$$l = \frac{w + l}{Q}\lambda, \quad \text{or } l = \frac{w\lambda}{Q - \lambda}$$

and substituting this value in the above expression we obtain

$$s = \frac{m}{w}(Q - \lambda) + \lambda$$

Example.—Suppose a piece of silver weighs

	mgr.
in air	$m = 24312$
in water at 19°·2	$= 21916$
the apparent loss of weight in water, w, =	2376

The uncorrected specific gravity will therefore be—

$$\frac{m}{w} = \frac{24312}{2396} = 10{\cdot}147$$

We obtain the corrected value, taking $Q = 0{\cdot}99843$ for 19°·2, from Table 4—

$$s = 10{\cdot}147\ (0{\cdot}99843 - 0{\cdot}0012) + 0{\cdot}0012 = 10{\cdot}120.$$

III. Correction for Changes of Temperature of Observations with the Pyknometer or Immersed Weight.

If the temperature of the specific gravity bottle alters between the different weighings, a further correction becomes needful on account of the expansion of the water and glass. The weight of water which the flask would contain at any given temperature can be calculated in the following manner from the weight, taken once for all, of the bottle when full of water at any one temperature.

Let us call the temperature and density of the water at the time the weighing was performed t_0 and Q_0 (Table 4), the weight of water p_0, and the corresponding quantities for another temperature t, Q, p. p is to be calculated.

(1.) If only the most considerable correction—viz. that depending upon the expansion of the water—is to be considered, we have

$$p = p_0 \,.\, Q/Q_0 \text{ or approximately } p = p_0 + p_0(Q - Q_0)$$

(2.) If we have regard to the expansion of the flask, we know that the volume is greater in the proportion $1 + 3\beta(t - t_0)$ for t than for t_0 where 3β denotes the coefficient of cubical

expansion of the glass. This may usually be taken as $\frac{1}{40000}$. We have therefore

$$p = p_0\{1 + 3\beta(t - t_0)\}\frac{Q}{Q_0} = p_0 + p_0[3\beta(t - t_0) + Q - Q_0]$$

(3.) These directions have special importance in determinations of the density of small solids, since by not applying the corrections we might be led to altogether erroneous results. The apparent weight w of a volume of water equal to that of the body is obtained by the following formula:—

$$w = m + P_0 - P + (P_0 - \pi)[Q - Q_0 + 3\beta(t - t_0)]$$

In this formula

m = the weight of the body in air;
P_0 = the weight of the bottle when full of water;
P = the weight of the bottle with the substance, and filled up with water;
π = the weight of the empty flask (need only be approximate).

Further, the temperature and density of the water are—

t_0, Q_0 at the weighing with water alone;
t, Q „ with water and the substance;
3β = the coefficient of cubical expansion of the glass.

Proof.—It is obvious that, if p_0 and p be the weights of the water at t_0 and t, $p = p_0\{1 + 3\beta(t - t_0)\}Q/Q_0$. This expression can be simplified by bearing in mind that 3β, the coefficient of cubical expansion of the glass, is always a very small number; and further, that Q and Q_0 only differ very little from 1. For by writing $1 + (Q - 1)$ for Q, and $1 + (Q_0 - 1)$ for Q_0, we obtain by formula 8 (p. 11)—

$$\frac{Q}{Q_0} = \frac{1 + (Q - 1)}{1 + (Q_0 - 1)} = 1 + (Q - Q_0)$$

By formula 7, therefore, the above expression becomes

$$p = p_0[1 + 3\beta(t - t_0) + Q - Q_0] = p_0 + p_0[3\beta(t - t_0) + Q - Q_0]$$

i.e. the expression given under (2).

The glass with the water would therefore, at the temperature t, weigh

$$P_0 + (P_0 - \pi)[3\beta(t - t_0) + Q - Q_0]$$

But when the body, of the weight m, has been introduced into the vessel, the weight w of water has overflowed, and the whole now weighs P. Obviously therefore

$$P + w = P_0 + (P_0 - \pi)[3\beta(t - t_0) + Q - Q_0] + m$$

from which the expression for w given under (3) follows.

It will be seen at once that the weight π of the empty vessel need only be determined approximately, for it only occurs multiplied by a factor of the dimensions of a correction.

(4.) *Correction for changes of temperature* in determining the density of fluids with the specific gravity bottle or the glass weight. Both corrections are plainly identical.

Let the weight of water in the bottle (or the loss of weight of the glass weight) be p_0 at temperature t_0.

At the temperature t let the weight of the fluid (or the loss of weight of the glass weight) be m.

Calculate out $w = p_0\,[1 + 3\beta\,(t - t_0)]$ and finally if Q_0 be the density of the water at t_0 (*cf.* II.)

$$s = \frac{m}{w}(Q_0 - 0{\cdot}0012) + 0{\cdot}0012.$$

When s differs but little from 1 the 0·0012 disappears.

IV. Reduction to a Normal Temperature.

s is determined for temperature t. For a solid body t is of course its temperature in the water.

Hence the density S at any other temperature T is found with the aid of the cubical coefficient of expansion α (or 3β, Table 9)

$$S = s[1 + \alpha(t - T)]$$

Most fluids have an irregular expansion which must be taken from formulæ or tables. Let the volumes of the same quantity of fluid at T and t be V and v

then $$S = s\frac{v}{V}$$

(*Cf.* Table 9; also Hofmann-Schädler, *Tabellen für Chemiker;* Gerlach, *Salzlösungen;* Landolt u. Börnstein, Tabellen 30, etc.)

14.—DETERMINATION WITH THE VOLUMENOMETER (Say; Kopp).

A volume of air V is shut off at the atmospheric pressure of H mm. of mercury (height of the barometer). If this measured volume V be increased by v cc. and the diminution of pressure h mm. be observed, we have

$$V = v\frac{H-h}{h}$$

If, on the other hand, V be diminished, and an increase of pressure h be observed—

$$V = v\frac{H+h}{h}$$

When the volume of the empty vessel has been found, the body is placed in it and the same process is gone through. The difference of the values found is the volume of the body; the density is therefore the weight (in grammes) divided by this difference.

In order that the results may be satisfactory, v and h should not be too small.

Any alteration of the temperature of the enclosed air, through the influence of neighbouring bodies, etc., must be avoided during the experiment.

(*Cf.* Myers on Volumenometer devised by Stroud. *Phil. Mag.* xxxvi. 195.)

15.—CALCULATION OF THE DENSITY OF THE AIR OR OF A GAS FROM ITS PRESSURE AND TEMPERATURE.

Let d_0 be the density (as compared with water) under the pressure of 760 mm. of mercury and at 0° (Table 1); then for the pressure H (**20**) and temperature t, the density d is, according to Marriotte's and Gay-Lussac's laws—

$$d = \frac{d_0}{1+0{\cdot}00367t} \cdot \frac{H}{760}$$

The values of the expressions $1+0{\cdot}00367t$ and $H/760$ may be found by Table 7. 0·00367 is nearly 11/3000 or 1/273.

The density of dry atmospheric air for 760 mm., and 0° in latitude 45° is, according to Regnault

$$\lambda_0 = 0{\cdot}001293$$

To the temperature t and pressure H, reduced to 0° and corrected for difference of gravity to 45° latitude, corresponds therefore the density

$$\lambda = \frac{0{\cdot}001293}{1 + 0{\cdot}00367t} \cdot \frac{H}{760} \tag{1}$$

Table 6 is calculated from this formula for convenience of reduction.

The density of other gases at H and t as compared with that of water is most readily obtained by multiplying λ by the density of the gas as compared with air, which is given in Table 1.*

The more easily liquefied gases have a somewhat greater coefficient of expansion. With increasing pressure or decreasing temperature the coefficients slightly increase.

If a volume of gas v is measured over a liquid (*e.g.* water) so that the volume v is saturated with the vapour of it, we obtain according to Dalton's law the pressure of the *dry* gas by subtracting from the total pressure the tension of the vapour of the liquid. In the case of water see Table 13; for other fluids see Landolt and Börnstein, Table 22, etc.

Density of Moist Air.—The density of water vapour is about $\frac{5}{8}$ths of that of air of the same pressure and temperature. The density of moist air is therefore found, if the tension (pressure) of the vapour of water in it is e, (**28**) by subtracting $\frac{3}{8}e$ from the total pressure (height of the barometer), and using the value of H thus corrected in the formula given above, or in finding the density from Table 6.

If e is not known the air may be assumed, on an average, to be half saturated with water vapour. This assumption is very nearly made, for ordinary temperatures, by taking for H

* The density of the air depends, of course, on its composition. If, for instance, in a room the amount of carbonic acid were five times as great as the normal quantity, λ_0 would become about $= 0{\cdot}001294$. For most purposes even such change of composition would be of no consequence. The greatest differences from the mean density in the open air were found $= \pm 1/3000$ (Jolly).

the undiminished height of the barometer, but making the formula

$$(2) \qquad \lambda = \frac{0{\cdot}001295}{1 + 0{\cdot}004t} \cdot \frac{H}{760}$$

Moist air may be as much as 1 per cent lighter than dry air under the same conditions of temperature and pressure.

16.—Determination of Vapour Density.

By vapour density is meant the density of a vapour (or gas) referred to dry atmospheric air of the same temperature and pressure (tension). The vapour density is equal to the molecular weight divided by 28·9; thus for water H_2O it is $18/28{\cdot}9 = 0{\cdot}623$.

In chemical work it is usual to take as the unit, not air, but a gas of half the density of Hydrogen; *i.e.* the density as referred to air is multiplied by 28·9 (since air is 14·44 times as dense as hydrogen). The vapour density is then simply equal to the molecular weight.

A. *By Weighing a Known Volume of Vapour* (Dumas).

For this purpose a light glass flask of from $\frac{1}{10}$ to $\frac{1}{4}$ liter capacity is used, *e.g.* a glass bulb with a tube blown on and drawn out, after drying, to a point with an opening of about a square mm.

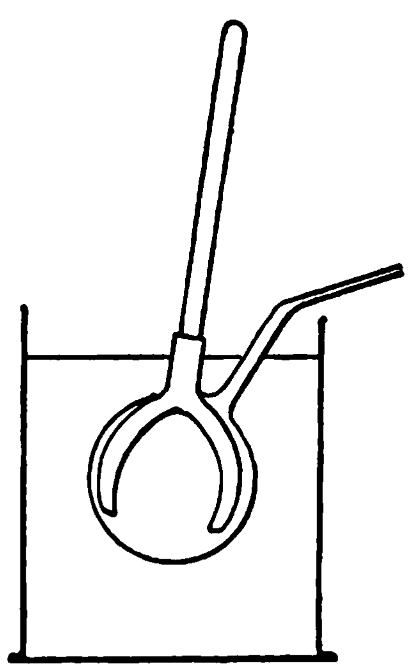


Fig. 3.

The dried vessel is weighed. Then a few grams of the fluid under examination are introduced. For this purpose the glass is warmed and the fluid sucked up as it cools. The bulb is then provided with a suitable holder (Fig. 3) and placed in a bath so that the open point reaches out of it; the bath is heated at least 10° to 20° above the boiling-point of the fluid. When all the fluid is evaporated, the bulb is hermetically sealed by drawing out the point in the blow-pipe flame. The temperature of the bath and the height of the barometer are read off at this instant.

After removal from the bath some of the fluid condenses in the bulb. By inverting the bulb, the drop of liquid is allowed to run to the point to make sure that this lets no air in. The cooled and well-cleaned bulb is then, with the piece of the point removed in sealing, again weighed, observing the height of the barometer and the temperature of the balance case. Lastly, the point of the bulb is held under water, from which the air has been removed by boiling or by the use of the air-pump, or else in mercury; a file-mark is made on it, and it is broken off below the liquid which rises into the bulb. The filled bulb, and the point which has been broken off, are again weighed. (See III. as to residual air.)

Instead of the glass bulb a vessel provided with two tubes closed by ground stoppers may be used with advantage, as it can be more conveniently dried and filled, and serves for repeated experiments (Pawlewski).

We call

(1.) m, the weight of the bulb when full of air;
(2.) m' „ „ „ vapour;
(3.) M „ „ „ water (or mercury);
(4.) t and b the temperature of the vapour, and the height of the barometer at the moment of sealing.
(5.) t' and b', the temperature in the balance-case and the height of the barometer at the weighing with the vapour. If the tension e of the aqueous vapour in the balance-room be observed (28), $\frac{3}{8}$ of its value must be subtracted from b' (but not from b) (15).
(6.) λ' the density of the air, which may be determined from t' and b' by the foregoing article, or taken from Table 6.

I. *Approximate Formula.*—The vapour density is, if water was used—

$$d = \left(\frac{m' - m}{M - m}\frac{1}{\lambda'} + 1\right)\frac{b'}{b}\frac{1 + 0{\cdot}00367t}{1 + 0{\cdot}00367t'}$$

If mercury be used instead of water $13{\cdot}56/\lambda'$ must be written instead of $1/\lambda'$.

Proof.—If D and L denote respectively the weight of the vapour and of the air contained by the bulb, we plainly have $D - L = m' - m$, therefore $D = m' - m + L$. The vapour density d

would, if the vapour and the air had both had the temperature t' and pressure b', be given simply by

$$d = D/L = (m' - m)/L + 1$$

or since $L = \lambda'(M - m)$ by

$$d = \frac{m' - m}{M - m} \cdot \frac{1}{\lambda'} + 1$$

But since the vapour at the time of sealing had temperature t and pressure b instead of t' and b' this d must be further multiplied by

$$\frac{b'}{b} \cdot \frac{1 + 0{\cdot}00367t}{1 + 0{\cdot}00367t'}$$

II. *Accurate Formula.*—Regard is had to the expansion of the glass and water with the temperature, and to the loss of weight of the water when weighed in air. (We neglect the change of the loss of weight of the bulb and weights with the change of temperature and pressure; and that the drop of the fluid which remains in the bulb has a density differing from that of water.)

In addition to the notation above (1) to (6), we have—

(7.) Q = the density of the water used for weighing (Table 4), [Mercury, Tables 1 and 9].

(8.) 3β = the coefficient of cubical expansion of the glass; average $3\beta = 1/40000 = 0{\cdot}000025$.

Then—

$$d = \left(\frac{m' - m}{M - m} \frac{Q - \lambda'}{\lambda'} + 1\right) \{1 - 3\beta(t - t')\} \frac{b'}{b} \frac{1 + 0{\cdot}00367t}{1 + 0{\cdot}00367t'}$$

Proof.—As in **13**.

III. It frequently happens that the atmospheric air is not completely expelled by the boiling of the substance in the bulb, which is known by the bulb not becoming completely full when the point is broken under water. If we do not intend to take account of this, the globe must be filled up with the wash bottle before the weighing, and the calculation proceeded with according to the preceding formulæ. Otherwise the bulb must, after breaking off the point, be immersed so far that the inner and outer surfaces stand at the same height, and weighed filled to that extent. Then the rest is filled with water, and the weight M determined. We will put

(9.) The weight of the bulb partially filled with water (or mercury) $= M'$.

Then the vapour density is—

$$d_0 = \frac{(m' - m)\frac{Q}{\lambda'} + M - m'}{(M - m)\frac{b}{b'}\frac{1 + 0{\cdot}00367t'}{1 + 0{\cdot}00367t}\{1 + 3\beta(t - t')\} - (M - M')}$$

(*Cf.* R. Kohlrausch, *Praktische Regeln zur genaueren Bestimmung des specifischen Gewichtes.*

Proof.—The volume of the included air-bubble, deduced from the weights M and M', is, at the time of filling, $= (M - M')/(Q - \lambda')$; it was therefore at the time of sealing

$$v = \frac{M - M'}{Q - \lambda'} \cdot \frac{b'}{b}\frac{1 + 0{\cdot}00367 \,.\, t}{1 + 0{\cdot}00367 \,.\, t'}$$

The expression for d calculated above is therefore the density of a mixture of the volume v of air, and $V - v$ of the vapour; and if we call the density of the pure vapour d_0,

$$Vd = v + (V - v)d_0$$

from which

$$d_0 = (Vd - v)/(V - v)$$

Here if for d we substitute the value found by II., for v the value found above, and put

$$V = (M - m)/(Q - \lambda')\{1 + 3\beta(t - t')\}$$

we obtain after some transformations, with the aid of the approximation formulæ (p. 10) the formula given first.

Example.—The following data are given (the weights expressed in grams)—

$m = 29{\cdot}6861$ (air).
$m' = 29{\cdot}8431$ (vapour).
$M = 142{\cdot}41$ (completely with water).
$M' = 141{\cdot}32$ (partially with water).

Further

$b = 745{\cdot}6$ mm. $t = 99^\circ{\cdot}5$ (at the sealing).
$b' = 742{\cdot}2$ mm. $t' = 18^\circ{\cdot}7$ (at the weighing of the vapour).
$e = 9{\cdot}4$ mm.

The temperature of the water used for the weighing $= 17^\circ{\cdot}4$ whence (by Table 4) $Q = 0{\cdot}9988$.

We find (**15**) $\lambda' = 0{\cdot}001182$ (not taking account of e);
$\lambda = 0{\cdot}001176$ (taking account of e).

The accurate formula III. gives (taking account of e) the vapour density 2·777, II. gives 2·755, I. gives 2·765. Neglecting e makes the numbers 0·006 greater.

The density referred to Hydrogen = 2, or the molecular weight of the vapour is therefore (p. 55)

$$\mu = 2{\cdot}777 \,.\, 28{\cdot}9 = 80{\cdot}3$$

The expression $1 + 0{\cdot}00367t$, which occurs frequently, is found in Table 7. If this be not used, it is more convenient to put $\frac{272{\cdot}5 + t'}{272{\cdot}5 + t}$ instead of $\frac{1 + 0{\cdot}00367t'}{1 + 0{\cdot}00367t}$

B. *By Measurement of the Volume of Vapour from a Weighed Quantity of Fluid (Gay-Lussac, Hofmann).*

A small quantity of the fluid of which the vapour density is to be determined is introduced into a thin-walled bulb of glass, or, still better, a very small (from 0·1 to 0·2 c.c.) flask with a ground stopper, and its weight determined. The bulb and its contents are then placed in a glass tube full of mercury, dry and free from air (**19**), inverted in a vessel of mercury. The tube is graduated in cubic centimeters from the closed end, or in mm. simply, which can afterwards be calculated into volumes (**19**) remembering that the glass expands 0·000025 for each degree. If the fluid be very volatile, the bulb (or stopper) bursts as it rises, in which case the tube must be held sloping, so as to be completely full of mercury, to avoid breakage. The upper part of the tube is now heated in a suitable vapour bath (water, amyl alcohol 137°, aniline 183°, see Table 16A; an inverted condenser must be used with the two last fluids) to a temperature at least 10° to 20° above that at which the whole of the liquid is evaporated.

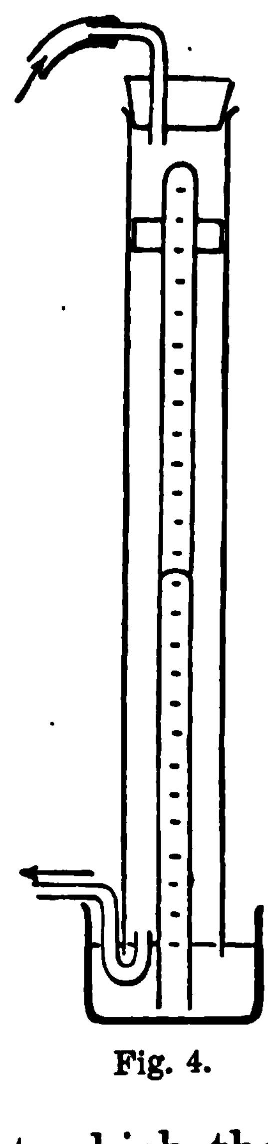


Fig. 4.

If now we call

m, the weight of the evaporated substance (in grms.);

v, the volume of the vapour (in c.c.); if v_0 is the volume at 15° of the part of the glass tube filled with vapour $v = v_0\{1 + 0{\cdot}000025(t - 15)\}$.

t, the temperature of the vapour;

b, the height of the barometer in the room;

h, the height of the mercury over which the vapour is, above that in the bath; b and h being reduced to 0° and in delicate measurements to 45° latitude (**20**).

e, the tension of the vapour of mercury for the temperature t (Table 14);

the desired vapour density (see beginning of article) is

$$d = \frac{m}{v} \cdot \frac{1 + 0{\cdot}00367 \,.\, t}{0{\cdot}001293} \cdot \frac{760}{b - h - e}$$

$$\text{or } d = \frac{m}{v} \cdot \frac{1}{\lambda}$$

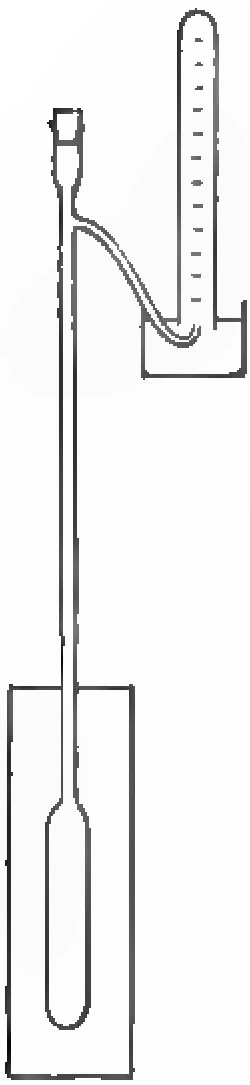


Fig. 5.

in which λ can be taken from Table 6 for temperature t, and height of barometer $b - h - e$.

C. *Displacement Methods.*

(1.) *By Displacement of Air* (V. Meyer).—The volume of the vapour of a weighed small quantity of the substance is determined by the quantity of air displaced during the evaporation. A glass (or, for high temperatures, porcelain) bulb with a vertical tube, provided with a narrow side tube of about 1 mm. diameter, is well dried, a pad of asbestos placed in the bottom, and brought to the requisite temperature above the boiling-point of the substance under experiment in an air bath, or in a bath of vapour either of water, aniline 183°, sulphur 448°, or in a bath of melted paraffin up to about 350°, or lead over 330° (see Table 16A). The experimenter then waits until the temperature becomes constant, *i.e.* until no more bubbles of air are driven out of the side tube, the end of which is under water.

The substance is weighed; when necessary, in a little wire basket or little glass tube; if it is fluid in a little flask or in a little completely filled sealed glass globe which is broken by the expansion of the substance. The cork is lifted, the substance rapidly dropped into the bulb, and the cork immediately replaced. At the same instant a measuring glass filled with water is pushed over the end of the gas delivery tube and receives the air expelled by the evaporating substance. It is essential that the operation should be performed quickly, the temperature of the bath should therefore be considerably above the boiling-point of the substance (long-continued expulsion of air may indicate a decomposition of the substance). The volume of air in the measuring cylinder is now read off.

Calling

m, the substance employed in grms.;
v, the measured volume of air in c.c.;
t, the temperature of the room;
H, the pressure on the air to be measured in mm. of mercury at 0°;

the required vapour density is—

$$d = \frac{m}{v} \frac{760}{H} \frac{1 + 0{\cdot}004t}{0{\cdot}001293} = 587800 \frac{m}{Hv}(1 + 0{\cdot}004t)$$

The vapour has expelled a volume of air, which, under similar conditions, possessed the same volume as the vapour. Consequently the weight of the substance m divided by the weight of this volume of air gives the required vapour density. The measured air, however, weighs $v\frac{0{\cdot}001293 \, . \, H}{(1 + 0{\cdot}004t) \times 760}$, which at once gives the above expression. The factor 0·004 is taken instead of the coefficient of expansion 0·00367 to allow for the moisture of the air. This at ordinary temperatures corresponds approximately to the assumption that the air in the bulb is two-thirds saturated, while that in the tube over the water is completely so.

(*Cf.* V. Meyer, *Berichte der Deutschen Chem. Ges.*, 1878, p. 2253, and 1888, p. 2018).

The pressure H is naturally that of the barometer b, diminished by the pressure h of the water column below the measured air, calculated into millimeters of mercury.

Therefore

$$H = b - \frac{h}{13{\cdot}6}$$

If, before reading off the volume, the measuring tube be plunged into the water, so that the inner and outer water surfaces are at the same level, H equals simply the barometric pressure b.

For more accurate measurement and calculation, the volume v' of the substance thrown into the bulb must be considered. If we further suppose that the bulb be filled with *dry* air, then with sufficient accuracy

$$d = \frac{587800}{\dfrac{v}{1 + 0{\cdot}00367t} + \dfrac{v'}{1 + 0{\cdot}00367t'}} \cdot \frac{m}{H - e}$$

where e is the tension of aqueous vapour for the temperature t (Table 13) and t' the temperature of the bath, which need only be approximately known.

(2.) *By Displacement of a Metal.*—The evaporating body of known weight (see B and C, 1) displaces a fluid which itself has only a small vapour tension (at low temperatures Mercury, Hoffmann, see Table 14; at higher temperatures Wood's Metal, V. Meyer).

Calling

m, the weight of the evaporated substance;
M, s and M', s', the weight and the specific gravity of the metal before and at the time of the displacement;
t, the temperature of the room;
T, the temperature of the bath—*e.g.* 448° for boiling sulphur;
b, the height of the barometer;
h, the height of the fluid metal in the other limb of the apparatus;
the density is

$$d = \frac{m}{\dfrac{M}{s} - [1 + 0{\cdot}000025(T - t)] - \dfrac{M'}{s'}} \cdot \frac{760(1 + 0{\cdot}00367T)}{\left(b + \dfrac{hs'}{13{\cdot}56}\right)0{\cdot}001293}$$

for the last factor see Table 6. The specific gravities are, at temperature t,

Mercury	$13{\cdot}60\ (1 - 0{\cdot}00018t)$
Wood's metal	$9{\cdot}6\ \ (1 - 0{\cdot}00009t)$

17.—Determination of the Density of a Gas.

A. *By Weighing.*

In order to determine the density of a permanent gas, a glass globe with a tube, melted on (best closed by a stop-cock), is filled with the gas by first filling the globe with mercury, inverting in it a mercury-trough, and displacing the mercury by the ascending gas. The globe is closed and weighed (m'). Then the gas is displaced by a sufficient current of air (air of the balance-room, not dried), and the globe weighed open (m). Lastly, weighing the globe filled with mercury gives the weight M. As in **16**, A, let b and t represent the height of the barometer and the temperature at the instant of shutting in the gas, where the height of the remaining column of mercury has already been subtracted. t' and b' are similar data for the weighing of the globe full of gas. The density of the gas is then calculated according to formula I. or II., pp. 56, 57.

A small quantity of mercury left in when filling with gas is without influence if it is left unaltered in all the weighings.

If a sufficiently large quantity of the gas is provided the glass globe with two necks described in **16**, A, or the pyknometer figured in **13**, can be used and the air driven out by a steady current of the gas. If the gas is heavier than air the openings are held upwards and *vice versa.* If the globe filled with air weighs m, filled with gas m', with water (or mercury) M, the calculation is simply by the formula given under **16**, A. If the experiment is so arranged that the temperature and atmospheric pressure is the same at the time of both fillings and weighings, the density of the gas is simply found (formula in **16**, A) as

$$d = \frac{m' - m}{M - m} \cdot \frac{Q - \lambda}{\lambda} + 1$$

The variations of the atmosphere are eliminated by using as a tare for the globe, a closed one of the same size. If then dry air is used in filling the globe the corrections on account of b, e and t disappear.

B. *By observing the Time of Escape of a Gas* (Bunsen).

The densities of gases are approximately in the inverse ratio of the squares of the velocity with which they escape under equal pressure through a fine opening in a diaphragm. If therefore the time which a given volume of gas requires to escape is compared with that required by an equal volume of air under the same conditions the ratio of the times squared gives the density of the gas.

According to Bunsen, a glass cylinder with a stop-cock closed at the upper end with a piece of thin metal foil melted on in which a very fine hole has been made, is taken and filled successively over pure mercury (**19**) with air, and the gas to be examined, both dried and rendered free from dust by passing through cotton wool. A stop-cock with two openings is convenient for the filling. The cylinder is now plunged so deeply into the mercury that the float is invisible and the stop-cock is opened. The position of the surface inside the cylinder cannot be read off directly on account of the opacity of the mercury, but is observed by the aid of a float which is carried by the mercury within the cylinder and is provided with several easily visible marks, one at the upper end, the others some cm. above the lower end. The times are observed at which these marks appear above the surface of the outer mercury. Some marks just above those observed should be noted to give warning of the emergence.

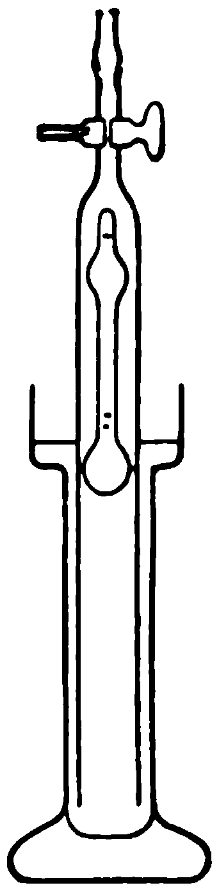


Fig. 6.

Example.—

	Air	Carbonic Acid.
Emergence of the upper mark . .	14·3 sec.	42·5 sec.
„ „ lower „ . .	51·2 „	87·8 „
	36·9 „	45·3 „

The density therefore of the carbonic acid referred to air is

$$d = \left(\frac{45 \cdot 3}{36 \cdot 9}\right)^2 = 1 \cdot 507$$

referred to Hydrogen = 2, or Molecular weight = 1·507 . 28·9 = 43·6. See Bunsen's *Gasometry*.

MEASUREMENT OF SIZE.

18.—Measures of Length.

The Divided Rule.

1. *Direct Reading.*—With regard to this most usual measurement it is only the principal source of error, the parallax in the reading off, which needs notice. In order to bring the object to be measured into the same plane as the divisions it is, in addition to the ordinary means, specially suitable to use a transparent rule which can be placed with the graduations on the object.

A graduation on the surface of a mirror before which the object can be fixed avoids the parallax if the image of the eye which is being used coincides with the point to be read off. When the graduation is not on a mirror a small piece of looking-glass can be laid upon it.

The most complete freedom from parallax is obtained by observing the graduations through a telescope pointing perpendicularly to the measure, and movable parallel with it.

2. *Comparator.*—The comparator which is most simple in use carries on the normal measuring rod a sliding piece with microscope attached. Such a comparator may generally be constructed out of a cathetometer by substituting a microscope for the telescope, and fixing the rod in a horizontal stand. The condition of accurately parallel sliding movement in the sliding piece must be the more strictly fulfilled the greater the distance of the object to be measured from the graduation.

The measurement is independent of the exact parallelism of the movement if the object and the normal measuring rod

are interchanged under the comparator. For this purpose either the microscope movable over the graduation may be used, or a pair of fixed microscopes clamped to a bar. Deviations from exact divisions of the measuring rod may in both cases be determined by means of a micrometer of known scale value in the eyepiece of the microscope (see below); in the first case in addition with the vernier attached to the sliding piece.

In accurate measurements with a vernier it must not be overlooked, firstly, that the vernier itself must be verified; secondly, that from a table of the errors of the measure the error of that particular division at which the vernier stands must be taken.

Verniers which are divided to read to tenths have either $\frac{9}{10}$ or $\frac{11}{10}$ of the interval of the main divisions as their unit.

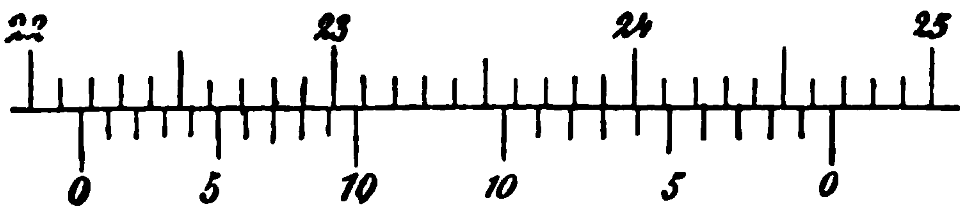


Fig. 7.

Both of the verniers figured read 0·7. With verniers reading to $\frac{1}{10}$ mm. it is easy to estimate $\frac{1}{100}$ mm. from the distances of successive divisions.

3. *Dividing Engines.*—These can be used for measuring, especially for small lengths, if a microscope with cross wires in the eyepiece is fixed either on the carriage or the stand. The value of the screw thread is determined on a divided rule. To avoid "lost time" it is well always to work in one direction.

4. *Microscope.*—For small lengths a microscope with an eyepiece micrometer is the best to use. With a micrometer divided on glass laid on the stage, the values of the scale divisions of the eyepiece micrometer are first determined and then are used in a manner easily seen. The eyepiece micrometer may itself consist of a scale on glass or of a cross of wire movable by means of a micrometer screw, the displacement being read off on the divided head of the instrument.

It must not be overlooked that a constant magnifying power requires the relative position of the eyepiece micrometer and the objective to be invariable.

5. *Verification of a Divided Measuring Rod.*—If a measur-

ing rod already verified is available the problem of preparing a table of corrections for another rod is already solved above. If not, the nominally equal divisions are compared with one and the same length a, and determined by means of their mutual ratios. Both the comparators mentioned under No. 2 afford the means for accurate measurements of this nature. If the length L contains n subdivisions $a_1, a_2 \ldots a_n$, and it was found that $a_1 = a + \delta_1$, $a_2 = a + \delta_2$, etc... to $a_n = a + \delta_n$—put

$$\delta = \frac{1}{n}(\delta_1 + \delta_2 + \ldots \delta_n)$$

Then

$$a_1 = \frac{1}{n}L - \delta + \delta_1, \quad a_2 = \frac{1}{n}L - \delta + \delta_2, \text{ etc. etc.}$$

In order, in a great number of comparisons, to avoid the accumulating of errors, the larger as well as the smaller divisions are compared. For instance, in the case of a rod divided into mm., all the dm., all the cm., and all the mm. are compared—the last by No. 4. Each larger division is, in the calculation, first treated as a whole with respect to its subdivisions.

(For more accurate methods see Benoit, *Trav. et Mém. du Bureau internat. des poids et mesures*, vol. ii. p. C. 35, etc.; or Benoit, *Constr. des Étalons prototypes de résistance Électr.*, p. 71, etc.)

6. *Preparation of Divided Measures.*—The ordinary dividing engine consists of a carriage with engraving tool movable on a screw of known thread. To eliminate "lost time" each division is approached from the same direction. For wood, ivory, and soft metal, a steel graver serves, in other cases a diamond tool must be used. Glass is usually coated when warm with a thin layer of wax, in which when cool the divisions are made. The divisions are etched in with solution of hydrofluoric acid applied with a brush, or by the vapour of hydrofluoric acid (from fluorspar and sulphuric acid) in a lead trough.

Bunsen copies the graduation of an original measuring rod by means of a long lath with two points fixed in it. The original rod and that to be divided are fixed in the same

straight line, one of the points is set in the division and a short mark made in the wax with the other.

Contact Measures.

The instruments known under the name of lever and screw gauges serve to measure with greater or less accuracy the distance from each other of the two end surfaces of a body. In the use of them the principal thing of consequence is to prove the correctness of the zero-point or apply to the measurements the necessary corrections.

7. *Spherometer.*—The screw is used in the spherometer for delicate measurements of thickness. The principle of this instrument, which takes many forms in construction, is that at a definite height the point of the measuring screw comes into contact with a suitable object. The contact is recognised by the fact that the instrument no longer stands steady, but rocks upon the movable point or is easily turned round it.

If a glass plate is placed between the point and the support, the contact is sharply evidenced by the displacement of the interference bands between the plate and the support, specially plainly seen in the light of the sodium flame.

Or else a contact lever or contact level is the object with which the contact is made. In this case the adjustment is always to the same reading of the pointer, or the same position of the bubble.

When a body, the thickness of which is to be measured, is introduced between the point and the support, the screw must be turned back through a distance equal to this thickness in order that the contact may be again produced as before. The number of whole revolutions of the screw is counted or is read off on the scale by the side; the divisions of the screw-head, which revolves in front of a fixed index, furnish the fractions of the revolutions. The number of revolutions multiplied by the length of the screw-thread gives the thickness required. The diameter of wires, etc., is measured between edges or plates.

On the measurement of radius of curvature see **43**, I.

8. The contact-comparator for the comparison of larger measures has also contact levers in combination with a micro-

meter screw. The methods of measurement are in principle simple.

Corrections.

9. *Temperature.*—The lengths of bodies vary with the temperature. If β is the coefficient of expansion of a rod (**26**; Table 9), l the length at a temperature t, l' at t'

$$l' = l[1 + \beta(t' - t)]$$

If, with a measure of which the coefficient of expansion $= \beta_0$ and the normal temperature t_0, a length was found apparently $= l$ at temperature t, the true length is

$$l_0 = l\{1 + \beta_0(t - t_0)\}$$

see also the example to **3**.

10. *Moisture of the Air.*—Wood and ivory are also dependent on moisture in the matter of their form. In the direction of the grain, maple and pine are little affected, mahogany, oak, and walnut on the other hand considerably.

11. *Flexure.*—The length of the axis of a rod alters but little by moderate bending. The distance of points out of the axis may however be increased or diminished in an easily perceptible manner. It is usually advisible when a measuring rod is used in a horizontal direction to support it in two places each $\frac{2}{9}$ of the length from the end. The rod is also best kept in the same manner.

18A.—Cathetometer (Dulong and Petit).

The cathetometer serves to measure vertical distances. A horizontal telescope which can rotate round a vertical axis is movable by a sliding carriage on a vertical measuring rod. The cathetometer can not be used with confidence at a great distance from the object to be measured on account of inaccuracy of the adjustment, the bending of the measuring rod, and the large errors produced by tremors. The adjustment of the instrument is performed as follows:—

1. The telescope is movable round its own axis. The cross wires are first of all so adjusted that during this rotation

a point under observation does not appear to change its position with regard to the cross wires.

2. The similarity of the cylinders in which the tube of the telescope turns is proved by the level placed on it, which must give the same reading when the telescope is reversed in its bearings and the level replaced in its original position.

3. The axis of rotation of the cathetometer is made vertical by so regulating the foot-screws that the bubble of the level keeps a constant position with regard to the graduations on it during a revolution. On the best order for the adjustments and the correction of the level itself see **88**.

4. The vertical position of the measuring rod is sufficiently recognised, or in case of need corrected, by means of a plumb line.

5. The horizontal position of the axis of the telescope is assured when by (1) the optical axis corresponds with the axis of figure; and by (2) the bearing cylinders of the telescope are equally thick and the level on the telescope shows the same reading on reversal. Or since by (4) the axis of rotation is vertical; a point is observed, the instrument turned through 180°, and the telescope reversed; the previously observed point must then have the same position with regard to the cross wires.

6. The assumed parallel movement of slide and carriage is proved by the constant reading of the level. In case of need, especially when the height to be measured is a great distance from the instrument, the position of the telescope must be corrected for each observation by means of the level, or the measurement is repeated with the telescope reversed and turned through 180°, and the mean taken of the two readings.

7. On temperature corrections see **18**, 9.

18B.—OPHTHALMOMETER (Helmholtz).

This instrument is used for the measurement of small distances; it consists of two equally thick glass plates, placed near each other in front of the object-glass of a telescope. They can be simultaneously rotated round a common axis through equal angles but in opposite directions. The amount

of rotation is read off on a divided circle. In the zero position both plates lie in the plane perpendicular to the optic axis of the telescope. The two points the distance of which is to be measured are observed through the telescope, and the glass plates rotated until the images of the points, displaced by the refraction in the obliquely placed glasses, coincide. The distance of the object from the instrument is without influence on the results.

If ϕ = the angle of rotation from the zero position;
a = the thickness of the plates;
n = the refractive index of the glass;
the linear distance e of the two points is

$$e = 2a.\sin\phi\frac{\sqrt{n^2 - \sin^2\phi} - \cos\phi}{\sqrt{n^2 - \sin^2\phi}} \quad \text{or} \quad 2a\sin\phi\left(1 - \frac{\cos\phi}{\sqrt{n^2 - \sin^2\phi}}\right)$$

The constants of the ophthalmometer $2a$ and n may either be directly measured for the glass plates used (**18**, particularly **18**, 7, and **39A** and **40**, II.) Or, better, the adjustment is made on various intervals in a mm. scale; of these determinations, of course, at least two are requisite. If more are made $2a$ and n are deduced by the method of least squares (**3**, III. and IV.)

Since the absolute symmetry of the instrument cannot be assumed it is well to perform each measurement twice with reversed inclinations of the glass plates, and take the mean of the two values of ϕ.

19.—Determination of Capacity by Weighing.

Measuring vessels as met with in commerce are often very incorrect,* weights are much more reliable.

A volume is determined or corrected by weighing with water or mercury.

If the fluid used to fill the vessel has in air the net weight m gr. and the density s (**12**) the volume in c.c. is

$$v = \frac{m}{s}\left(1 + \frac{\lambda}{s} - \frac{\lambda}{\delta}\right)$$

* Glass-blowers usually deduce the c.c. from the apparent weight of water at 15° weighed in air. From this cause the liter is 1·9 c.c. too large!

if λ is the density of the air (0·0012; **16** and Table 6) and δ that of the weights used (see **10**, and Table 8).

For s in the case of water see Table 4. In the case of water weighed against brass weights we have with sufficient accuracy $\frac{\lambda}{s}-\frac{\lambda}{\delta}=0{\cdot}00106$ (p. 39), and we may calculate the density simply as $v=m(2{\cdot}00106-s)$.
A gram of water at 15° as determined by the balance corresponds to the volume 1·0019 c.c. For other temperatures the numbers, all reduced to the volume of the glass at 15°, will be found in the second part of Table 4.

In the case of mercury of temperature t the density is $s=13{\cdot}596(1-0{\cdot}000181t)$ and $\frac{\lambda}{s}-\frac{\lambda}{\delta}=-0{\cdot}000055$.

A gram of mercury at 15° has the volume 0·07375 c.c.

Purification of Mercury.

Mercury is dried on the surface with blotting-paper, more completely by warming in a clean iron or porcelain dish to about 150° with stirring. Dust is removed by filtration, most simply through an ordinary filter (taking more than one where the weight is great), with one or more fine holes in the point. Grease is removed by shaking with some caustic soda or potash solution or benzol and alcohol, followed by repeated shaking with water. Foreign base metals and oxide are got rid of by shaking the mercury with dilute nitric acid or solution of ferric chloride or bichromate of potash, again followed, of course, by thorough washing, by shaking with water. Or the mercury is passed in fine streams through a column of 1 to 1½ meter of these liquids and then through water. The tube is bent at the bottom so that the mercury may retain the column of fluid by its pressure, and any excess flow away. Noble metals are removed by distilling the mercury, best *in vacuo*, *i.e.* in apparatus similar to a barometer.

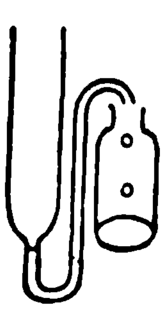


Fig. 8.

See *e.g.* Weinhold, *Carl. Rep.* ix. 69, 1873; xxiii. 791, 1887. Leonh. Weber, *ib.* xv. 1, 1879; Dunstan and Dymond, *Phil. Mag.* xxix. 367, 1890.

Even very slight impurity is rendered evident by the fact that the mercury does not flow cleanly, and after long standing shows a cloudy or sluggish moving surface.

If the volume of the vessel at temperature t_0 is to be

calculated from the observed volume at temperature t, we have

$$v_0 = v\{1 + 3\beta(t - t_0)\}$$

if β is the coefficient of linear expansion of the vessel. For glass we have as an average $3\beta = 1/40000$.

It is obvious that in the case of vessels to be used as measures of volume by pouring out the contents the net weight of the fluid used for filling must be got by subtracting from the gross weight the weight of the *wetted* vessel.

The influence of the meniscus is most practically eliminated by always reading off in the same way; as a rule it is best to use the horizontal plane touching the meniscus. The constancy of direction of sight in the observation which is necessary to prevent parallax is obtained by using a telescope which is movable along a vertical rod; or more simply by always bringing the point to be observed to coincide with the same distant point.

In the calibration with mercury of a graduated cylinder it is possible, according to Bunsen, to replace the repeated weighing of the mercury by a more simple method. A small glass tube is prepared, sealed at one end and ground flat at the open end, which, closed by a glass plate, contains a known volume of mercury (sp. gr., see previous page). The measure of mercury is poured into the vessel to be calibrated, the position of the surface observed, and the process repeated again and again. The influence of the meniscus may be determined by pouring a dilute solution of corrosive sublimate on the mercury by which the surface is rendered flat (Bunsen's *Gasometry*).

19A.—Calibration of a Narrow Glass Tube.

The tube, cleaned and well dried by means of a current of air, is laid horizontally on a measuring rod (with a mirror to avoid parallax in the readings) and a thread of pure clean mercury introduced which can be moved along the tube. This can be effected either by inclining the tube or with the aid of a piece of indiarubber tube fixed on the end of the tube; the end of the indiarubber tube is closed with one hand and the thread of

mercury moved backwards or forwards by pressing or releasing the pressure with the other.

In order to divide the tube into equal volumes the thread is brought into successive positions nearly continuous with each other and its length noted at each. These lengths then correspond to equal volumes. In the division into many subdivisions the errors of reading accumulate. It is in this case better to combine observations with longer and shorter threads. In order, for example, to divide into twenty-five parts it is well to begin with a thread one-fifth of the length of the tube and then subdivide the resulting divisions with a thread one-fifth of the length of the first.

The results are embodied in a table or a curve on paper ruled in squares from which intermediate values can be interpolated.

For more delicate methods of calibration see Marek, *Carl. Rep.* xv. 300, 1879; Benoit, *Trav. et Mém. du Bureau internat. des Poids et Mes.* ii. 35.

Absolute Calibre.—A mass of mercury of m mgr. (**11** and **19**) has at temperature t the volume $v = m(1 + 0{\cdot}000181t)/13{\cdot}596 = m(1 + 0{\cdot}000181t) \,.\, 0{\cdot}07355$ cubic mm. The mean sectional area of the measured length is, if l mm. is the length of the thread, v/l square mm.

The radius r of a circular tube of section q is found directly as $r = \sqrt{q/\pi}$.

Section Found by Weighing a Tube.—If a circular tube of external radius R, length l, and specific gravity of the substance of the tube s, has the weight m, the internal sectional area $= R^2\pi - m/ls$. This method is available for tubes with thin walls. For ordinary glass s may be taken $= 2{\cdot}5$.

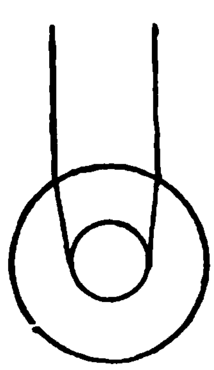


Fig. 9.

Optical Determination of Internal Diameter.—If a tube is examined from some distance the apparent internal diameter is equal to the true diameter multiplied by the refractive index of the tube, assuming that the internal and external surfaces are circular in section. In the case of ordinary glass, therefore, the true diameter is two-thirds of the apparent.

The diameter can thus be determined approximately by means of a measure held in front, or more accurately by the ophthalmometer (**18**B). To prevent deception by reflections the tube is held in front of a uniformly lighted surface.

By the Capillary Rise.—If a fluid of sp. gr. s and capillary constant a $\left(\text{water } 7{\cdot}8, \text{ alcohol } 2{\cdot}3\ \frac{mg}{mm}\right)$ rises to the height H in the well-wetted tube, the radius of the tube is $r = 2a/Hs$ (see **37**B).

Temperature.—For 1° the diameter of the glass increases by $\frac{1}{120000}$, the area by $\frac{1}{60000}$.

ATMOSPHERIC PRESSURE.

20.—HEIGHT OF THE BAROMETER.

By the height of the barometer is understood, generally, the height of a column of mercury at 0°, which balances the atmospheric pressure. On account of the variation of gravity which may amount to $\frac{1}{2}$ per cent, the condition must be added, for accurate physical purposes, that the force of gravity acting on the mercury shall be that experienced at the level of the sea in latitude 45° (see 5).

That the barometer is free from air is easily known by the sharp ring with which the mercury strikes the glass on inclining the instrument. The presence of vapour of water above the mercury is indicated by the deposit on the glass on inclining the tube in the case of large quantities only. If the height of the mercury can be again read after diminishing the space above the mercury, either by pouring in more mercury into the open arm or by plunging the tube more deeply into the cistern, a smaller reading will be obtained in presence of air or water vapour.

On the vernier see **18.**

The readings of the barometer require the following corrections.

1. *Temperature of the Mercury.*—Mercury expands 0·000181 of its volume for each degree of temperature. Therefore, if l be the height of the barometer as read off at the temperature t, its value, reduced to 0°, is (**4**, Example 2)—

$$b = l - 0{\cdot}000181 \, . \, l \, . \, t$$

It is commonly sufficient to use for l in the correction

member the value 750 mm., and perform the correction by subtracting 0·135 . t mm.

2. *Temperature of the Scale.*—The length of the scale must also in accurate measurements be reduced to its normal temperature t_0, by the addition of $\beta(t-t_0)l$, where β denotes the coefficient of expansion (lineal) of the material of the scale (0·000019 for brass, 0·000008 for glass). If, as is usually the case, the normal temperature be 0°, the height of the barometer completely corrected for temperature becomes—

$$b = l - (0{\cdot}000181 - \beta)lt$$

The correction amounts therefore

$$\text{for a brass scale to } -0{\cdot}000162 \,.\, l \,.\, t$$
$$\text{for a glass scale to } -0{\cdot}000173 \,.\, l \,.\, t$$

values for which are to be found in Table 11.

3. *Capillary Depression of a Cistern Barometer.*—To correct for this there must be added, to the reading given by the top of the meniscus, the amount in Table 15, corresponding to the internal diameter of the tube and the height of the meniscus of mercury.

The only completely certain method is to use a so-called *normal* barometer, the wide tube (25 mm.) of which prevents any noticeable depression. The comparison of another instrument with a *normal* barometer eliminates the capillary depression of the former.

4. *Tension of Mercury Vapour.*—At a high temperature t this causes a slight depression (Table 14), which is corrected with sufficient accuracy by adding $0{\cdot}001t$ to the observed height.

5. *Variation of the Force of Gravity.*—The pressure of one and the same column of mercury is at different places proportional to the force of gravity which obtains there. Calling the force of gravity at the level of the sea in latitude 45°, g_{45}; and that in latitude ϕ and at a height above sea level of H meters g,

$$g_{45} = 980{\cdot}6 \text{ cm./sec.}^2$$

and

$$\frac{g}{g_{45}} = 1 - 0{\cdot}0026 \cos 2\phi - 0{\cdot}0000002H$$

The observed height of the barometer must therefore be multiplied by this expression, the last term of which, however, need only be used in the case of very considerable altitudes, to obtain the height which the same atmospheric tension would produce in latitude 45° at sea level.

The height of the barometer, b, expressed in cm., thus reduced gives the pressure in grams, *i.e.* the weight per square centimeter as $b \cdot 13{\cdot}596$. The pressure in the absolute cm.-grm.-sec. system, *i.e.* the force in dynes per square centimeter, or in [cm.$^{-1}$ grm. sec.$^{-2}$] (Appendix 6) is $b \cdot 13{\cdot}596 \cdot 980{\cdot}6 = b \times 13332$.

A normal atmosphere corresponds to the pressure

$$76 \cdot 13{\cdot}596 = 1033 \text{ grm.-wt./sq. cm., or } 76 \cdot 13332 = 1013200 \; [\text{cm.}^{-1} \text{ grm. sec.}^{-2}] \text{ or dynes.}$$

The remarks under 1 to 5 are applicable to all accurate measurements of pressure by columns of mercury.

6. An aneroid barometer is verified or provided with a table of corrections by comparison with a mercury barometer. The instrument is placed, for example, under the air pump, connecting with the receiver a sufficiently wide glass tube in which mercury is sucked up, and the height of the column thus raised subtracted from the barometer reading—applying the corrections 1 to 5. The temperature correction of an aneroid must be determined empirically.

21.—Measurement of Heights by the Barometer.

If the height of the barometer be observed at the same time at two different stations, or if the mean height of the barometer at each be known, the difference in height of the stations may be obtained by the following rules. We denote by

b_0 and b_1 the two barometer readings [reduced to the same temperature, and, if necessary, corrected for the tension of the mercury vapour (previous article); as well as for any difference of scale in the two instruments];

t_0 and t_1 the temperature of the air at the two stations;

h the required difference of height in meters;

and for convenience calling, further,

t = the mean of the temperatures of the air at the two places,—

therefore $$t = \tfrac{1}{2}(t_0 + t_1)$$

I. It is usually reckoned that

$$h = 18420 \text{ met. } (log.\ b_0 - log.\ b_1)\ (1 + 0{\cdot}004 \,.\, t)$$

from which, for differences of height not exceeding 1000 meters, we may obtain the more convenient approximation—

$$h = 16000 \text{ met. } \frac{b_0 - b_1}{b_0 + b_1}\ (1 + 0{\cdot}004 \,.\, t)$$

II. In these formulæ gravity at sea level in latitude 45° and a mean amount of moisture in the air are assumed.

If now

ϕ be the geographical latitude;
H the mean height above the sea of the two places in meters (the influence of this is scarcely ever perceptible);
finally e_0 and e_1 the tensions of the vapour of water at the two stations (**28**);

and calling for shortness

$$k = \tfrac{1}{2}\left(\frac{e_0}{b_0} + \frac{e_1}{b_1}\right)$$

the difference of height is

$$h = 18405 \text{ met. } (log.\ b_0 - log.\ b_1)\ (1 + 0{\cdot}00367t)$$
$$.\ (1 + 0{\cdot}0026\ cos.\ 2\phi + 0{\cdot}0000002H + \tfrac{3}{8}k)$$

The logarithms in this formula are the common Briggs's logarithms.

For convenience of carrying, the height of the barometer in measurement of heights is also deduced from the boiling-point of water. Tables 13A and 13B give the corresponding boiling-points and pressures. Since 1 mm. of pressure corresponds to $\frac{1}{25}$ of a degree, it follows that very sensitive, accurately verified thermometers, as well as the greatest care, must be employed in the temperature determination (**22**) if we wish to arrive at a tolerably accurate result.

Proof of the Hypsometric Formula.—The density of atmospheric air (**15** and **20**) in latitude ϕ, the height H, with the height of barometer b, the temperature t, and the tension e of the aqueous

vapour; calling, for shortness, $0{\cdot}0026 \,.\, cos\ 2\phi = \delta$, $0{\cdot}0000002 = \epsilon$, and $0{\cdot}00367 = a$, is

$$\frac{0{\cdot}001293}{1 + at} \cdot \frac{b - \frac{3}{8}e}{760}(1 - \delta - \epsilon H)$$

Now the density of mercury at 0° is 13·596; it follows, if the increase of height dH diminish the height of the barometer b by db (*i.e.* dH and db are the heights of the columns of air and mercury respectively which are *in equilibrio*)—

$$-db = \frac{0{\cdot}001293}{13{\cdot}596\,.\,760}\,(b - \tfrac{3}{8}e)\,\frac{1 - \delta - \epsilon H}{1 + at}\,dH$$

Here, besides b, we have e and t varying with H, but according to an unknown law. Hence we take for t the constant mean value, and put e in a constant ratio to the height of the barometer, $e = kb$. If, then, we calculate out the numerical factor, and consider the small quantities $\frac{3}{8}k$, δ, and ϵH, according to p. 10, as "corrections," we may write—

$$-7993000\,(1 + at)\,(1 + \delta + \tfrac{3}{8}k)\,\frac{db}{b} = (1 - \epsilon H)\,dH$$

Integrating now between the limits b_0 and b_1 on the left-hand side, and H_0 and H_1 on the right, we have—

$$7993000\,(1 + at)\,(1 + \delta + \tfrac{3}{8}k)\,(log.\ b_0 - log.\ b_1) = (H_1 - H_0)\,[1 - \tfrac{1}{2}\epsilon(H_1 + H_0)]$$

the logarithms being natural logarithms.

Finally, putting *natural log.* $b = 2{\cdot}3026\ log._{10}\ b$, and considering $\frac{1}{2}\epsilon\,(H_1 + H_0) = \epsilon H$ as a correction, we obtain

$$H_1 - H_0 = h = 18405000\ \text{mm.}(log.\ b_0 - log.\ b_1)(1 + at)(1 + \delta + \epsilon H + \tfrac{3}{8}k)$$

The approximation formula for unknown humidity is got by assuming the air half saturated, and neglecting the influence of the aqueous vapour on the density and the coefficient of expansion (15).

The approximation formula under I. without logarithms, which is applicable for small differences of height, is only the differential formula given above, which omitting the corrections for gravity and humidity and the sign becomes

$$7993000\,(1 + at)\,db/b = dH$$

dH is the difference in height; for the difference of the height of the barometer db we write $b_0 - b_1$ and for the mean height b, $\frac{1}{2}(b_0 + b_1)$. Reducing mm. to meters removes the 3 ciphers, and 7993 is taken as 8000 for a round number. Hence the approximation formula follows at once.

HEAT.

22. Mercury Thermometer—Freezing-point and Boiling-point.

Temperature is scientifically defined by means of the expansion of a perfect gas (hydrogen) by making equal increments of temperature correspond with equal increments of volume or pressure in the gas. In addition, the two fixed points for water,—the freezing and boiling points under the normal pressure,—are taken as the fundamental data of the thermometric scale. We use the centigrade scale, *i.e.* we call the temperature of melting ice 0° and the temperature of water boiling under a pressure of 760 mm. (**20**) 100°.

The scale of the mercury thermometer which is generally used has not an exact agreement with that of the air thermometer because neither the mercury nor the glass expands uniformly (see **24**). The first problem is to make the mercury thermometer correct in its own readings.

A. *Freezing-point.*

The thermometer is plunged into clean melting snow or clean (washed) finely crushed, or, better, scraped or filed ice; moistening with distilled water is advisable. The column of mercury must be as completely as possible immersed in the ice; thermometers must be immersed beyond the zero and only during the reading off be as far as necessary freed from ice, not removed from it, since by this means warm air reaches the mercury. Special attention is required to any melting of the ice away from the bulb of mercury which may introduce considerable errors in warm places.

The point at which the mercury column stands when the thermometer has taken up the temperature of the ice corresponds with the temperature 0°.

The warmer the surrounding air is the more carefully must the foregoing precautions be observed.

B. Boiling-point.

The thermometer is placed in the steam from water boiling vigorously in a metal vessel or in one of glass with pieces of metal in it. The temperature of the steam is found from the pressure under which the water boils—*i.e.* from the height of the barometer corrected (**20**)—by the aid of Table 13B. Without tables, the boiling-point may be determined to within $\frac{1}{100}$ of a degree for any pressure between 715 and 770 mm. by the formula—

$$t = 100° + 0°{\cdot}0375\,(b - 760)$$

The bulb of the thermometer must not dip into the boiling water, but must be slightly above the surface. If the water is not pure, the thermometer must be protected from the spray. Here also the whole column of mercury should be exposed to the steam. The opening for the escape of the steam must be so wide that there is no additional pressure in the vessel, or this additional pressure is measured by a water manometer communicating with the interior of the vessel, and $\frac{1}{14}$ of the height of the column of water raised added to the height of the barometer. The flame should be kept at some distance from the parts of the vessel which are not in contact with water. In such a vessel as is here figured the bulb may be farther from the surface of the water than the distance given above.

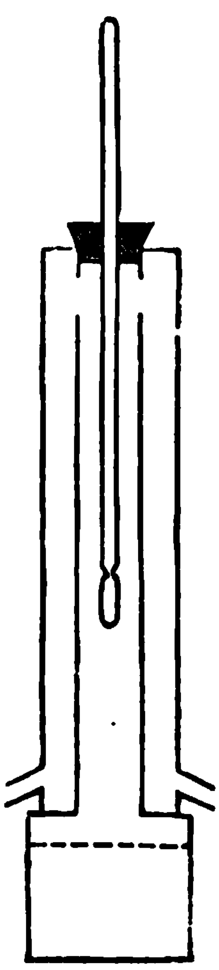


Fig. 10.

The thorough heating of the thermometer requires some time, specially in the case of enclosed thermometers. The reading should not be made until the position is constant.

In exact determinations the reading is best made with a telescope. The thermometer is adjusted in a vertical position by means of a plumb-line, window-frame, or

something similar, and the telescope is placed at the height of the division to be read. A simpler means to avoid parallax is to attach a small slip of looking-glass to the thermometer by two indiarubber rings and hold the eye so that its image corresponds with the top of the column. In using a lens to read the degree, the curvature of the mark when at any other height gives the means to recognise the proper position.

Example.—The reduced height of the barometer was 742 mm. The thermometer in the steam stood at 98°·8. The boiling-point is found from Table 13B to be 99°·33 (from the formula given above, $100 - 0{\cdot}0375 \times 18 = 99{\cdot}33$). It follows that 100° is denoted by the division $98{\cdot}8 + 0{\cdot}67 = 99{\cdot}47$, or the correction of the thermometer reading $= +0{\cdot}53$.

C. Alteration of the Fixed Points.

(1.) In the case of long columns the position of the thermometer with respect to the vertical has, on account of the pressure of the mercury, a small influence on the reading of the mercury. This influence must of course be determined empirically. If it is found that a thermometer gives for the same temperature a reading δ higher in the horizontal position than when vertical, the correction to the vertical position amounts when the inclination is ϕ to $\delta \sin \phi$.

(2.) On account of the gradual contraction of the blown glass the two fixed points of new thermometers first of all rise, and by nearly equal amounts. Long heating, say to the boiling-point, with slow cooling appears to accelerate the contraction.

(3.) Since the expansion of the glass, after a subsequent heating of the finished thermometer, has an effect which takes time to completely disappear, each heating leaves behind an enlargement of the bulb, and thereby a lower position of the mercury, a so-called depression of the zero which sensibly disappears usually in the course of some hours, but, after long heating, only after months.

The amount of this depression depends upon the amount and duration of the previous heating. After long-continued heating the depression reaches a quantity which is nearly proportional to the previous rise of temperature (Pernet).

Long heating to 100° produces depressions, varying with

the sort of glass, up to 1°. Even the temperature of a room or of the body is therefore not without perceptible influence.

If a thermometer that has been heated is placed in a constant low temperature, the enlargement of the bulb (the "fatigue" dilatation) begins at once to disappear; the thermometer therefore soon begins to slowly rise.

If the thermometer were kept some time in boiling water and then placed in ice, it would after a short time take up a minimum reading and then begin to slowly rise again. This lowest reading is called "the zero of maximum depression." It characterises a thermometer with as much definiteness as the freezing-point which would be shown after very long standing in the ice, and since this latter, in thermometers which had been considerably heated, would demand a very long time to attain, the observation of the zero of maximum depression offers advantages.

Glass like the Jena normal thermometer glass shows very slight depression.

(4.) Continuance in a very high temperature may be followed by a raising, in some cases considerable (up to 20°), of the fixed points. Thermometers for high temperatures should be, before use, heated for some days.

D. Definition and Calculation of Temperature.

We assume for this purpose an accurately calibrated thermometer (**23**). The ordinary definition of temperature makes the thermometer, at any temperature, come to rest. The zero is that point at which the mercury stands after remaining a long time in ice; from this point to that attained after long boiling there are 100 degrees, and the temperature scale is now simply calculated from equal volumes between these fixed points.

Against this must be urged that the interval between the boiling-point (100°) and that zero-point which the thermometer shows immediately after the determination of the boiling-point is constant, and is much more easily determined than the interval actually used, because the operations for this require a long time; further, the other temperatures which the instru-

ment has previously had influence the readings. On this account delicate thermometric measurements of recent date require the following definitions (Pernet):—

(1.) A degree is the one-hundredth part of the interval between the boiling-point and that freezing-point which is found immediately after the boiling.

(2.) Temperature t is always reckoned from that zero-point which is found, or would be found, immediately after the temperature determination. (The zero in this definition is therefore a variable point.) This depressed zero lies at about $d \,.\, t^2/100^2$ below that determined after a long rest, where d is that depression below the zero after standing, which is produced by long heating (say for half an hour) to 100°.

E. The Exposed Part of the Column of Mercury.

A considerable difficulty in the way of accurate measurement of temperature arises as a rule when any long portion of the column of mercury is not immersed in the space to be experimented on. Since the apparent coefficient of expansion of mercury in glass, *i.e.* the difference of the coefficients of expansion of volume of the two substances amounts to 0·000156, there must be added to the reading t of a thermometer

$$0{\cdot}000156a(t - t_0)$$

if t_0 is the temperature of the exposed part of the column of mercury and a the number of degrees occupied by it. The accurate determination, however, of the mean temperature of this exposed part is difficult.

(1.) A small auxiliary thermometer is taken and its bulb placed at about the middle of the exposed column; or better, several are placed at different points along it, and the temperature of the column determined by the auxiliary instruments.

(2.) The following procedure is more trustworthy (Mousson, Wüllner). The temperature of the room is taken as the temperature of the column, but the length of the column at this temperature is taken, not as the whole length exposed, but a constant quantity a determined as follows is subtracted from it. If the thermometer show in a warm bath of con-

stant temperature (such as the boiling-point vessel, p. 82) the degree T when it is completely immersed, and only t when A degrees are exposed, and if τ_0 be the temperature of the air; then plainly

$$a = A - \frac{1}{0{\cdot}000156} \cdot \frac{T-t}{t-\tau_0}$$

The quantity a thus found is therefore, in the use of this thermometer, always to be subtracted from the length a of the exposed mercury column, but the air temperature is taken as t_0.

(3.) On the use of a correction tube in the thermometer, see Guillaume, *Comptes Rendus*, cxii. 1, 87, 1891.

LITERATURE RELATING TO C. AND D.—J. Pernet, *Carl. Rep.* xi. 257, 1875; *Meteor. Zeitschrift*, 1877, pp. 129, 206; *Travaux et mémoires du bur. internat. des poids et mesures*, i. 1881. Also Thiesen, Grunmach, Wiebe, Weinstein, *Metronomische Beiträge*, No. 3, Berlin, 1881; Wiebe, *Zeitschrift für Instrumentenkunde*, 1888, p. 373; 1890, p. 207 (on alteration of freezing-point).

23.—CALIBRATION OF A THERMOMETER.

From the irregular section of the tube there arise in ordinary thermometers errors which, at high temperatures, sometimes amount to more than 10 degrees. We are to prepare, for a thermometer in which only a correct linear division and a scale nearly corresponding to the true temperatures are assumed, a table of corrections, by which the readings can be reduced to those of a normal thermometer—*i.e.* of one of which the 0 and 100 correspond to the freezing and boiling points, and of which all the scale divisions have equal volumes.

We must therefore calibrate the tube—that is to say, compare the volumes which correspond to the divisions of the scale at different places. For this purpose a thread of mercury, separated from the rest, is made use of (see also **19A**).

Separation of a Thread of Mercury of any Desired Length.—The thermometer is turned upside down, and a slight tap given against the end. Then either a thread will separate, or the whole of the mercury will flow down, separating from the

walls of the bulb at some point. The separation is usually determined by a microscopical air-bubble adhering to the glass, which expands to a larger size. If the mercury separates from the glass in the bulb, we try, by suddenly turning the thermometer upright, to make the bubble formed there rise to the opening of the stem; this can always be done, with patience. The mercury, then, divides at the opening of the tube.

Suppose the thread to be too long, say p degrees longer than was desired. The bulb is warmed while the thread is separated; the air is pushed forward by the rising mercury. Then the thread is made to run rapidly back to the rest of the mercury, and the position of its upper end is observed at the instant of meeting. The little bubble of air remains adhering to the glass at the point of the stem where the junction took place. The thermometer is now cooled p degrees, and again reversed and shaken, when a thread of the desired length is separated.

If, on the other hand, the thread be p degrees too short, it is united to the rest, the thermometer warmed p degrees, when the desired length can be separated.

Even if this manipulation should not succeed at first, it will always, on repetition, be possible to get a thread accurately of any length to the fraction of a degree. For very short threads, however, the process often fails; so that in such a case we must make use, as shown below, of combined observations with threads of different lengths.

Placing and Reading the Thread.—By gentle inclining and shaking, one end of the thread can be adjusted to any desired division with great accuracy. In accurate observations, especially with the telescope, it is sufficient to place it nearly on the division, and estimate the tenths of a degree at both ends of the thread. It is a matter of course that the observations may be repeated and made more accurate by taking means.

To avoid parallax when reading off, the thermometer is laid upon a piece of looking-glass and the eye so placed that its image coincides with the division to be read.

It is convenient to place the thermometer perpendicular to the line joining the eyes, so that there is no need to shut one

eye during the reading. Or a lens is fixed and the thermometer is pushed along parallel to its length. The greatest accuracy is secured by reading with the telescope.

Observation and Calculation.—The calibration may be executed in many ways. In every case it is advisable to completely arrange beforehand the plan of the reduction, because otherwise one might afterwards be led into complicated calculations. The calculation will always be simplified by making the freezing and boiling points the extremities of compared volumes. Observations, according to the following plan, are sufficient for ordinary purposes, and the more so because completely corrected mercurial thermometers may differ not inconsiderably on account of the sort of glass of which they are made (see **24**, end).

Let a be the interval by which we wish to calibrate, and let a divide 100 without remainder, then $n = 100/a$, a whole number. We separate a thread of about this length a; this we place successively at the marks of the graduation from near 0 to a, a to $2a$, and so on (see below as to very faulty thermometers). In each position let the thread occupy the following number of divisions:—

$a + \delta_1$ from the mark 0 to a
$a + \delta_2$ „ „ a to $2a$
. . . . .
$a + \delta_n$ „ „ $(n-1)a$ to 100

Let it have been further determined (**22**)

that the temperature 0° corresponds to p_0
„ „ 100° „ $100 + p_1$

The quantities δ_1 δ_2 . . . as well as p_0 and p_1 are therefore small numbers, expressed in scale-divisions and fractions, and may be either positive or negative.

If, then, we use the abbreviation—

$$s = \frac{p_0 - p_1 + \delta_1 + \delta_2 + \ldots + \delta_n}{n}$$

(the sum of δ being only taken between 0° and 100°) the correction-table of the thermometer is—

Division.	Correction.
0	$-p_0$
a	$s-p_0-\delta_1$
$2a$	$2s-p_0-\delta_1-\delta_2$
.	. . . .
ma	$ms-p_0-\delta_1-\delta_2-\ .\ .\ -\delta_m$
.	. . . . . .

Or again, the correction for ma being Δ_m, if that for $(m-1)\,a$ be Δ_{m-1}

$$\Delta_m = \Delta_{m-1} + s - \delta_m$$

The values under the heading "Correction" are therefore those numbers which must be added to, or, when negative, subtracted from the corresponding reading, in order to obtain the corresponding reading of a mercurial thermometer accurate as to calibration, zero and boiling point.

For the intermediate degrees, a table is interpolated in the usual manner, best graphically.

Proof.—The thread of mercury used for the observations, laid end to end n times, takes up the volume of the tube from division 0 to 100, increased by $\delta_1+\delta_2+\ .\ .\ +\delta_n$. But since 0° is at division p_0 and 100° at $100+p_1$, the increase of the volume of mercury from division 0 to division 100 answers to an increase of temperature of $100+p_0-p_1$, so that the increase of the volume equal to the length of the thread means an increase of temperature—

$$\frac{100+p_0-p_1+\delta_1+\delta_2+\ .\ .\ .\ .\ +\delta_n}{n} = a+s \text{ (see above).}$$

Therefore a rise of the mercury

from 0 to a corresponds to an increase of temperature $a+s-\delta_1$
„ a to $2a$ „ „ „ $a+s-\delta_2$
. . . . . .

and finally,

from division 0	Temperature increase.	
to a	$a+$	$s-\delta_1$
to $2a$	$2a+$	$2s-\delta_1-\delta_2$
. . . .		.
to ma	$ma+$	$ms-\delta_1-\delta_2-\ .\ .\ .\ \delta_m$

The expressions to the right of the stroke would be the thermometer corrections, if the division 0 also meant the temperature

0°. Since the temperature $-p_0$ corresponds to this, p_0 must be subtracted from each of them.

Example.—A thermometer graduated to the boiling-point of mercury is to be calibrated at intervals of 50°, which is enough for ordinary purposes. Here, therefore, $n = \frac{100}{50} = 2$. A thread of about 50° long was separated, and occupied the spaces—

from	0·0	to	50·9	$\delta_1 = +0{\cdot}9$
,,	50·0	,,	100·4	$\delta_2 = +0{\cdot}4$
,,	100·1	,,	150·3	$\delta_3 = +0{\cdot}2$
,,	149·8	,,	199·6	$\delta_4 = -0{\cdot}2$, etc.

In addition, the temperature 0° was found to be at the division $+0{\cdot}6$, and 100° at 99·7; therefore

$$p_0 = +0{\cdot}6,\ p_1 = -0{\cdot}3$$

Therefore—

$$s = \frac{p_0 - p_1 + \delta_1 + \delta_2}{n} = \frac{+0{\cdot}6 + 0{\cdot}3 + 0{\cdot}9 + 0{\cdot}4}{2} = +1{\cdot}1$$

The table of corrections is therefore—

Division.			Correction.
0	$-0{\cdot}6$	=	$-0{\cdot}6$
50	$1{\cdot}1 - 0{\cdot}6 - 0{\cdot}9$	=	$-0{\cdot}4$
100	$2{\cdot}2 - 0{\cdot}6 - 0{\cdot}9 - 0{\cdot}4$	=	$+0{\cdot}3$
150	$3{\cdot}3 - 0{\cdot}6 - 0{\cdot}9 - 0{\cdot}4 - 0{\cdot}2$	=	$+1{\cdot}2$
200	$+1{\cdot}2 + 1{\cdot}1 - 0{\cdot}2$	=	$+2{\cdot}5$, etc.

The correspondence of the calculated correction for 100 with the determination of the boiling-point furnishes a partial proof of the accuracy of the calculation.

Thermometer with Larger Corrections.—As will be seen the method assumes that the thermometer to be calibrated is not very irregular in the diameter of the tube. For we have not taken into account that δ_1, δ_2, etc., are actually not degrees of temperature but scale divisions, nor that many points of the scale were either not covered or were covered twice. The more inaccurate the thermometer the less would these simplifications be justified.

It will mostly suffice in such a case to proceed thus: Firstly, the freezing-point and boiling-point are determined. Then a thread is separated nearly of such a length that, laid n times end to end, it would just cover the space from the freezing-

point to the boiling-point. It is then observed in such positions, beginning with the freezing-point, that each nearly adjoins the previous one. The calculation is then just the same as before. The corrections given by the table, however, are not for the points o, a, $2a$, etc., but for the points near them at which the ends of the mercury thread were. The preparation of a convenient table of corrections is performed as before graphically.

Calibration by several Threads.—We do not always succeed in separating a thread as short as the interval a by which we wish to calibrate. We must then use several threads, the lengths of which are different multiples of a. By one thread ka in length we can compare the capacity of the tube the scale-space between 0 and a with that between ka and $(k+1)a$, and so on, by bringing the thread first between 0 and ka, and then between a and $(k+1)a$; for the volume which is left empty by moving the thread is equal to that which is freshly occupied at the other end a. For example, a thread of say 40° long can be used to compare 0 to 20 with 40 to 60.

In order, however, to refer all the divisions to a common measure more threads must be used of the lengths, *e.g.* $2a$ and $3a$. With these the volumes to be compared are all in the shortest possible manner referred to one and the same interval *e.g.* to the middle one, and then the table of corrections can be calculated out in the manner set forth above. An example will make this sufficiently clear.

Example.—A thermometer is to be calibrated for every 20 degrees from 0 to 100 by means of two threads of 40° and 60° long. We take the middle part, that from 40 to 60, as the unit volume with which we are to compare the others. The observations, therefore, are reduced to those numbers which a thread of mercury F, which exactly fills the space from division 40 to division 60, would have afforded. According to the above given notation, therefore (p. 88),

$$\delta_3 = 0$$

Now, let the column of about 40° in two positions occupy the spaces from $+0{\cdot}3$ to $40{\cdot}0$ and $20{\cdot}7$ to $60{\cdot}0$. The column F would therefore have extended from $+0{\cdot}3$ to $20{\cdot}7$; therefore, $\delta_1 = +0{\cdot}4$.

In just the same way we reduce the space from 80 to 100 to F by observations between 40 and 80, 60 and 100. Suppose it has been found

$$\delta_5 = -0{\cdot}7.$$

Now, we take a column 60° long, place it between 0 and 60 and 20 and 80. By this we get 60 to 80 in terms of 0 to 20, and, since the latter space has already been compared with 40 to 60, to F. Let the included spaces be—

0 to 60·2, and 20 to 79·6;
therefore 0 to 20 = 60·2 to 79·6.

But the column F is longer than 0 to 20 by 0·4; it would therefore have extended from 60·2 to 80;

$$\text{therefore } \delta_4 = -0{\cdot}2.$$

Finally, in the same manner let observations between 20 and 80 and between 40 and 100 have given—

$$\delta_2 = +0{\cdot}3.$$

Further, let the temperature 0° be at + 0·1, and 100° at 100·8; therefore—

$$p_0 = +0{\cdot}1, \quad p_1 = +0{\cdot}8.$$

The number of spaces compared between 0 and 100 is $n = 5$. From this we calculate (p. 88)—

$$s = \frac{+0{\cdot}1 - 0{\cdot}8 + 0{\cdot}4 + 0{\cdot}3 + 0{\cdot}0 - 0{\cdot}2 - 0{\cdot}7}{5} = -0{\cdot}18.$$

And the table of corrections is obtained by using the formula, $\Delta_m = \Delta_{m-1} + s - \delta_m$.

Division.	Correction.
0	− 0·10
20	− 0·10 − 0·18 − 0·4 = − 0·68
40	− 0·68 − 0·18 − 0·3 = − 1·16
60	− 1·16 − 0·18 + 0·0 = − 1·34
80	− 1·34 − 0·18 + 0·2 = − 1·32
100	− 1·32 − 0·18 + 0·7 = − 0·80

The last number is a proof of the correctness of the calculation.

Thermometer for Calorimeter.—In a thermometer for this use it is only the interval from 12° to 25° which has to be dealt with. In order to obtain this in a simple manner a thermometer has first of all the point 50 determined with a column of 50°, then with a column of half and a quarter the length the points

25 and 12·5. A point near 17 is obtained with a column of about 33° the length of which is determined by adjusting three times between 0 and 100, by measuring with it backwards from 50. In this manner 12·5, 17, and 25 are obtained. The calorimeter thermometer is then referred to the one thus calibrated by comparison.

For more delicate methods of calibration see v. Oettingen, *Über die correction des Thermometers*, Dorpat, 1865; Thiesen, *Carl. Rep.* xv. 285, 1879; Marek, *ib.* 300.

Comparison of two Thermometers.—The table of corrections for a thermometer may also be deduced by comparison with a standard thermometer. The two instruments are placed in a vessel, not too small, filled with fluid, and protected as much as possible by felt or otherwise from loss of heat. The bulbs of the two thermometers should be close to each other in the liquid, which should be set in motion by stirring before each reading. At high temperatures the comparison may easily, in spite of all precautions, be inexact (see also the end of **22**).

Greater certainty is attained by the use of a boiling fluid than by a bath of fluid. It is best used in a vessel with an inverted condenser, into which the two thermometers are introduced.

Thermometer with Nitrogen.—These are used so that the division of the mercury column at high temperatures may be avoided. No column can be separated in them. The verification is therefore confined to comparison with another thermometer of which the corrections are already known, or with an air-thermometer (**24**). The boiling temperatures of fluids with high boiling-points may also be used.

Many standard thermometers have a graduation calibrated, but otherwise arbitrary. If the temperatures 0° and 100° are at p_0 and p_1, a reading p of the thermometer corresponds to $100(p-p_0)/(p_1-p_0)$.

24.—Air Thermometer.

The use of the air thermometer rests upon the assumption that a perfect gas (hydrogen, approximately also dry air) expands, at constant pressure, proportionally to the rise of tem-

perature. The expansion amounts for each degree to 0·00367 of the volume at 0° (hydrogen 0·00366). If the volume is kept constant the pressure varies in the same proportion.

The simplest air thermometer (the very convenient form given to it by Jolly) depends upon the latter law. A glass globe of about 50 c.c. capacity filled with *dry* air, is in communication, by means of a capillary tube, with a vertical glass tube I., in which the air is confined by mercury. By the raising or lowering of the surface of the mercury in II., which is joined to I. by an indiarubber tube, the surface of the mercury in I. can be adjusted to a mark near the opening of the capillary tube. For very high temperatures vessels made of porcelain or platinum are used—air pyrometer.

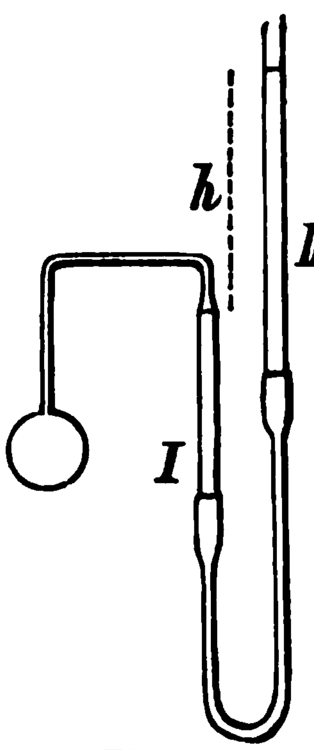


Fig. 11.

First Method.—To graduate the instrument, the bulb is surrounded with melting ice (**22A**), the mercury is adjusted, and the height of the barometer b_0, and the height h_0, of the mercury in II. above that in I. observed. We will call $b_0 + h_0 = H_0$, where h_0 is negative when the surface in II. is the lower. All the heights b and h must be reduced to 0° (**20**).

If, now, any other temperature t which is to be measured be communicated to the air in the bulb, the mercury adjusted to the mark, and the heights b and h be observed, calling $b + h = H$, we have—

$$t = \frac{H - H_0}{0{\cdot}00367\, H_0 - 3\beta H}$$

If the coefficient of cubical expansion of the glass is not known for the sort of glass used (**26**, II.) we may reckon $3\beta = 0{\cdot}000025$. Up to temperatures of about 60° we may calculate it with sufficient accuracy by the more convenient formula—

$$t = 275 \frac{H - H_0}{H_0}$$

It is here assumed that the volume of the capillary tube up to the mark to which the mercury is adjusted may be completely neglected in comparison with that of the bulb.

If not, the value of t, given above, must be multiplied by

$$1 + \cdot\frac{v'}{v}\cdot\frac{H}{H_0}\cdot\frac{1}{1 + 0{\cdot}00367\,t'}$$

where v = the volume of the bulb, v' that of the connections up to the mark, and t' = the temperature of the room.

The ratio $\frac{v'}{v}$ is found by weighing with mercury. If p be the weight of the mercury in the bulb alone, and P the weight when the apparatus is filled up to the mark—

$$\frac{v'}{v} = \frac{P - p}{p}$$

The measurement of the boiling-point of water (Table 13B) serves to verify the formula for t.

Proof.—The quantity of air remains constant. If v be the capacity of the bulb at 0°, d_0 the density of the air for 0° and 760 mm., the quantity of air is given, if we call $0{\cdot}00367 = a$, by—at the first observation —

$$\frac{d_0 H_0}{760}\left(v + \frac{v'}{1 + at'}\right)$$

at the second by—

$$\frac{d_0 H}{760}\left[\frac{v(1 + 3\beta t)}{1 + at} + \frac{v'}{1 + at'}\right]$$

By equating the expressions, dividing by $\frac{d_0}{760}$, and multiplying both sides of the equation by $\frac{1 + at}{v}$, we get—

$$H_0(1 + at)\left(1 + \frac{v'}{v}\cdot\frac{1}{1 + at'}\right) = H\left(1 + 3\beta t + \frac{v'}{v}\cdot\frac{1 + at}{1 + at'}\right)$$

or separating t—

$$t\left[aH_0 - 3\beta H - \frac{v'}{v}\frac{a}{1 + at'}(H - H_0)\right] = (H - H_0)\left(1 + \frac{v'}{v}\frac{1}{1 + at'}\right)$$

From this we get the first of the expressions given above by putting $\frac{v'}{v} = 0$. In order to obtain the correction, we write the left-hand side of the equation—

$$t\,(aH_0 - 3\beta H)\left(1 - \frac{v'}{v}\cdot\frac{a}{1 + at'}\cdot\frac{H - H_0}{aH_0 - 3\beta H}\right)$$

In the multiplier of the small magnitude $\frac{v'}{v}$ we may neglect the $3\beta H$, which occurs in the denominator, in comparison with aH_0; and finally we get (Formula 8, p. 11)—

$$t = \frac{H - H_0}{aH_0 - 3\beta H}\left(1 + \frac{v'}{v}\cdot\frac{H}{H_0}\cdot\frac{1}{1 + at'}\right),$$

as was to be proved.

Second Method.—Instead of assuming the coefficients of expansion 0·00367 and 3β, it is better to determine the freezing-point and boiling-point of the instrument. If H_0 is the pressure for 0° and H_1 that for t_1 the temperature of the boiling water, the pressure H corresponds to the temperature t—

$$t = t_1\frac{H - H_0}{H_1 - H_0}\left[1 - \frac{H_1 - H}{H_0}\left(\frac{3\beta}{0{\cdot}00367} + \frac{v'}{v}\,\frac{1}{1 + 0{\cdot}00367t'}\right)\right]$$

Here 0·00367 and 3β only occur in correction terms.

Comparison of Mercury and Air Thermometers.—The air thermometer is now generally taken as the standard instrument. Mercury does not expand exactly uniformly compared with air. According to Regnault its volume, which between 0° and 100° increases by 0·01816 may be expressed at temperature t of the air thermometer as

$$* \; v_t = v_0(1 + 0{\cdot}000179t + 0{\cdot}000000025t^2)$$

or up to $t = 100$ by the expression frequently more convenient

$$\log v_t = \log v_0 + 0{\cdot}000078t.$$

It is unfortunately impossible simply to refer the mercury to the air thermometer by these expressions, because the expansion of the glass also is not uniform, and is very different according to the sort of glass. Almost all mercury thermometers give readings rather too high between 0° and 100° if the calibration

* According to Recknagel and Wüllner it is better to use an expression with four terms:

$1 + 0{\cdot}0001802.t + 0{\cdot}0000000094t^2 + 0{\cdot}00000000005t^3$ according to Recknagel.

$1 + 0{\cdot}00018116t + 0{\cdot}0000000116t^2 + 0{\cdot}000000000021t^3$ according to Wüllner.

Bosscha puts $\log v_t = \log v_0 + 0{\cdot}0000785t$. The mean coefficient of expansion of mercury between 0° and 100° is, according to Wüllner and Bosscha, rather greater than that used by Regnault, namely $= 0{\cdot}0001825$.

and the freezing and boiling points are correct, or at least the corrections of **22** and **23** have been applied. Up to 150° the variations remain as a rule smaller than 0°·5; up to 250° they may amount to 4°; and up to 350° to 10°.

If the difference of a corrected mercury thermometer from an air thermometer at 50° be observed as Δ, the difference δ for a temperature *t*, up to 120°, may, according to Bosscha, be calculated to a good approximation by the formula

$$\delta = \frac{\Delta}{2500} t(100 - t)$$

In this relation also the Jena glass gives good results.

According to Wiebe (*Zeitsch. für Instr. Kunde,* 1890, p. 245) the corrections of a mercury thermometer made of Jena glass, to the air thermometer are, in hundredths of a degree—

Temperature	−20°	0	+20	40	60	80	100	120
Correction	+15	0	−8	−11	−10	−5	0	+5

140	160	180	200	220	240	260	280	300°
+9	+10	+6	−4	−21	−46	−82	−130	−190

25.—Determination of Temperature with a Thermo-element.

In experiments where the great mass or the bulk of a mercurial thermometer prevents its use, the electromotive force, developed at the point of contact of two metals (bismuth —antimony; iron—german-silver; platinum—iron; platinum —palladium) by difference of temperature, may often be made use of. Two wires of equal length (*e.g.* iron and german-silver) are soldered together at one end, and at the other to copper wires. If the former soldering be placed at the point of which the temperature is to be measured, and the other two kept at a known temperature (say by ice at 0°), an electromotive force results. This is measured by connecting the ends of the copper wires with a galvanometer and observing the deflection.

For small differences of temperature (up to about 20°) the current strength may be taken as proportional to the difference of temperature. It is therefore only necessary to

measure the current strength for a known difference *once*, in order to deduce the temperature from any observation. A galvanometer of moderate resistance, reading by a mirror (**66**), should be used. It is well to use only copper connections.

For greater differences, or when the ordinary thermo-multiplier is used, in which the current strength cannot be calculated from the deflections, a table is constructed empirically by observing the deflections for certain known temperatures. From this a table for use is interpolated either by calculation or graphically.

A convenient form of the thermo-element is the following:—*a* and *b* are the iron and german-silver wires (for use in mercury, iron and platinum), which are passed through a cork into a small glass tube full of alcohol or petroleum, within which they are soldered to the copper wires, which are brought through the other cork. In the alcohol a small thermometer can be placed.

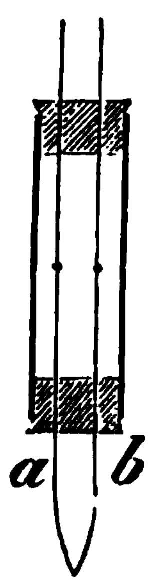


Fig. 12.

[For the measurement of high temperatures the couple platinum and platinum with 10 per cent of rhodium as suggested by Le Chatelier is very convenient. The galvanometer recommended is a modification of that of D'Arsonval, with a resistance of about 300 ohms, and the instrument is calibrated by observing the point of arrest when the junction is placed in a melted metal of known melting-point as it cools down. This point corresponds to the melting-point of the metal, and from a number of such points a curve may be constructed differing very little from a straight line above 300°. The couple may either be placed in a peculiarly shaped recess in the crucible of metal, or may be placed in a porcelain tube and plunged into the metal; see Roberts-Austen, *Proc. Roy. Soc.*, xlix. p. 347, 1891; *Proc. Inst. Mech. Engineers*, 1891, p. 543, and 1893, p. 102 (*Tr.*)]

26.—Determination of the Coefficient of Expansion by Heat.

The linear coefficient of expansion (β) is the increase of unit-length of the body for a rise of temperature of 1°; the

cubic (3β) the increase of unit volume by a rise of temperature of 1°. For liquids the expansion is always reckoned by volume.

I. *By Measuring the Length.*

If the length of a rod $= l$, and if it increase λ in length for a rise of $t°$, the coefficient of expansion $\beta = \frac{\lambda}{lt}$ (see also the example in **3**). The small expansions require delicate means of measuring them. If a contact lever be used, and the angle a through which it is turned be measured, $\lambda = r \sin a$, where $r =$ the distance of the point of contact from the axis on which the arm turns, and where also it is assumed that at one of the temperatures the lever arm is perpendicular to the direction of the rod.

The angle is conveniently measured by observing a scale reflected in a mirror fixed to the lever. We assume that at one of the observations the point at which a perpendicular from the mirror falls upon the scale is seen in the telescope, and that the distance between the scale and the mirror, expressed in scale-divisions as units, $= R$. If the motion of the image for the change of temperature amount to e scale-divisions, $a = \frac{1}{2} \tan^{-1} e/R$. Since when a is small we may put $2 \sin a$ for $\tan 2a$, we should have in this case $\lambda = \frac{e}{2} \cdot \frac{r}{R}$. (See also **3, 48,** and **49.**)

For large differences of temperature the expansion is no longer accurately proportional to the increase of temperature; the length at the temperature t is then taken as

$$l = l_0(1 + \beta t + \beta' t^2)$$

and the two coefficients β and β' determined from at least three observations. (See **3.**)

II. *By Weighing.*

It is very often needful to obtain an accurate knowledge of the coefficient of expansion of glass; this can be obtained by a process of weighing. A bulb with a drawn-out point is weighed,

and filled at two different temperatures with mercury (**19**). To fill the bulb, it is first warmed and the point plunged into mercury, of which, as the bulb cools, a quantity is drawn up into it. This is repeated until the bulb is completely full; at last, boiling the mercury. The bulb is plunged into warmed mercury and left there till it has cooled down to the lower temperature t. By weighing, the net weight p of the mercury is obtained. Then it is warmed to the temperature t', which causes a certain quantity of mercury to overflow, and the weight p' of the remainder is determined. Then the cubical coefficient of expansion is calculated thus—

$$3\beta = 0{\cdot}000182\frac{p'}{p} - \frac{1}{t'-t} \cdot \frac{p-p'}{p}$$

For the proof see next page.

If the weighings are performed with the bulb filled with water free from air at two temperatures t and t', the same formula holds; only we must use instead of 0·000182 the mean coefficient of expansion for water between t and t'. This is obtained from Table 5 as $\frac{1}{t'-t} \cdot \frac{v'-v}{v}$ or from Table 4 as $\frac{1}{t'-t} \cdot \frac{s-s'}{s'}$

Since, at high temperatures, the expansion of mercury, and still more that of water, far exceeds that of solids, very accurate determinations of temperature are necessary.

The coefficient of expansion may also be obtained from the determination of the densities s and s' at the temperatures t and t', viz.

$$3\beta = \frac{1}{t'-t} \cdot \frac{s-s'}{s'}$$

(**13**, 2 and 3 for solids and **13**, III.)

III. *Expansion of Liquids.*

(1.) A glass vessel with drawn-out point or in the form of the fig. on p. 45, when entirely filled, contains at the ordinary temperature t the weight of liquid p; and at the higher temperature t' the weight p'. If 3β is the cubical coefficient of

expansion of the glass (see above) the mean coefficient of expansion of the liquid between t and t' is

$$a = 3\beta\frac{p}{p'} + \frac{1}{(t'-t)} \cdot \frac{p-p'}{p'}$$

For if v and v' denote the volumes of the same quantity of liquid at the temperatures t and t'

$$a = \frac{1}{t'-t}\,\frac{v'-v}{v}$$

But plainly $p'/p = [1 + 3\beta(t' - t)]v/v'$, therefore

$$v'/v = p/p' + 3\beta(t'-t)p/p'$$

from which the formula easily follows.

(2.) If a glass weight be weighed in the liquid at two different temperatures, and p and p' are the respective losses of weight, the formula is the same as under 1.

(3.) A glass bulb with a narrow divided tube (dilatometer) is filled up to the tube and the position of the liquid observed at the temperatures t and t'. If the observed volumes are v and v' the mean coefficient of expansion is

$$3\beta\frac{v'}{v} + \frac{1}{v}\,\frac{v'-v}{t'-t}$$

The bulb is calibrated with mercury and the tube with a mercury thread, which is weighed (**19** and **19**A). It is still simpler to examine a fluid of known coefficient in the apparatus, and from this to calculate the ratio of the volumes.

26A.—Melting-Point or Freezing-Point.

By this name is known that temperature at which the solid and liquid portions of the substance can exist together (Table 16A). Mixtures of several substances, such as most fats, paraffin, or glass have usually no definite melting-point, but a temperature interval within which they soften.

Melting-point of a Body.—The method of making a determination varies much according to the nature of the substance, especially as to the temperature of the melting-point. For instance, a small quantity of the melted substance may be

drawn up into a glass tube drawn out to a fine point and allowed to solidify. The solid state will mostly be distinguished from the liquid by the drop becoming cloudy. The tube is placed along with a thermometer in a beaker filled with water, petroleum, paraffin, etc., which can be gradually warmed, stirring all the time, and the temperature is noted at which the drop becomes clear or movable. The first observation only serves for an approximation. Temperatures of solidifying are on the whole uncertain, and may be considerably lower than the melting-points.

Larger quantities of a substance are gradually heated along with a thermometer. At the point of melting the temperature remains for some time stationary.

The determination of high melting-points, for which the method given above, as also the use of the mercury thermometer, is out of the question, is a matter of difficulty. On air pyrometer, see p. 94. An approximation may be arrived at by means of the resistance of a platinum wire (**71**, *e*) or by the current from a thermo element of metals with high melting-points placed in contact with the substance under investigation, which can be melted on a thread of asbestos. A window of mica allows the interior of the furnace to be kept in sight. [See also *Tr.* note on the platinum, platinum-rhodium thermo couple, **25**.]

Freezing-point of a Solution.—The solution of a substance in any solvent lowers the freezing-point by an amount τ, which at first is proportional to the quantity of the substance dissolved. In the same solvent, further, the lowering of the freezing-point is inversely proportional to the molecular weight M of the substance dissolved (Rüdorff, Coppet, Raoult). If therefore p grms. of the substance be dissolved in 1000 grms. of the solvent and M be the molecular weight of the substance, and therefore $\mu = p/M$ the number of so-called gram-molecules dissolved in 1000 grms. of solvent, we have according to what has been stated above

$$\tau = C.\mu$$

where C is a constant depending on the solvent, and amounting to the following in the cases given.

Water	$C = 1 \cdot 87$	Freezing-point	0°
Benzol	5·0	„	+ 5°·4
Acetic acid	3·9	„	+ 16°·5
Formic acid	2·8	„	+ 8°

The freezing-point of the solvent used must of course be determined specially since the differences to be measured are small.

The molecular weight may thus be deduced from the lowering of the freezing-point. But it must be noticed that many bodies, especially electrolytes (salts, alkalis, acids) are excepted from the law. The actual lowering is in these cases greater than that calculated from the formula with the chemical molecular weights. This is explained on the assumption that such molecules are decomposed, "dissociated," in the solution. The excess of observed lowering of freezing-point divided by the calculated number would give the decomposition or "degree of dissociation" (Arrhenius).

The freezing-point is measured by gradually cooling the solution with constant stirring, a delicate thermometer being immersed in it. The temperature usually at first sinks some degrees below the freezing-point without solidification taking place; when the separation begins the temperature rises suddenly to the freezing-point, which is then read off.

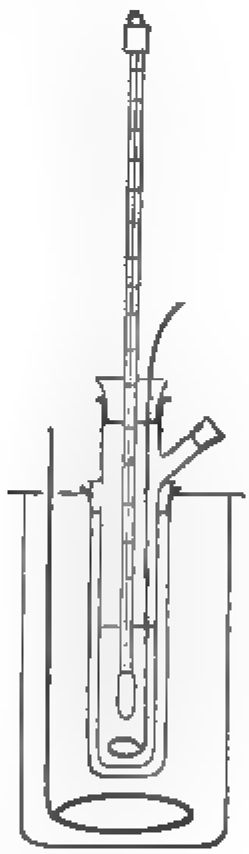


Fig. 13.

Beckmann's arrangement here figured facilitates this difficult measurement. An inner cylinder contains the solution (which is poured in through a side tube), a stirrer and the thermometer. From the column of mercury of this thermometer suitable portions can be detached according to the freezing-point of the solvent used. The lowering is of course reckoned from the position of the column of mercury in the freezing pure solvent. The substance to be dissolved is introduced by the side tube. This cylinder is fixed in a somewhat wider one, leaving an air space by which to separate the solution and the freezing mixture, which is placed in a

wider jar surrounding the whole. The temperature of the freezing mixture should not be too much below the freezing-point of the solution (3°), since otherwise this would appear too low.

(*Cf.* Beckmann, *Zeitsch. für Physik. Chem.* ii. pp. 638, 715, 1888.)

27.—Boiling-Point of a Fluid.

The boiling-point is the temperature of the vapour which rises from a fluid boiling under the pressure of 760 mm. of mercury at 0° (**20**). The direct readings of the thermometer require two corrections.

(A.) A part of the column of mercury is usually out of the vapour. For this correction, see **22**, E.

(B.) The boiling-point must be reduced to 760 mm. (**20**) from the actual height of the barometer b at the time of observation. It is indeed only very rarely that the rise of the boiling-point in proportion to the increase of pressure is known, which would be necessary to the accurate correction. But since the boiling-points of most fluids which have been investigated vary according to nearly the same law in the neighbourhood of 760 mm.—on an average, that is, this temperature increases by 0·0375, or $\frac{3}{80}$ of a degree, for 1 mm. increase of pressure—a probable correction may be applied by adding to the observed temperature

$$0{\cdot}0375\,(760 - b)$$

If the boiling-point of a solution be determined under height of barometer b (using an inverted condenser to prevent an alteration of the concentration), and if the vapour tension e of the solvent (Tables 13, 14) be known for the temperature at which the solution boils, the alteration ϵ of the tension produced by the dissolved substance is

$$\epsilon = b - e$$

(see **27**A).

The thermometer should only be immersed in the vapour of the fluid, except in the case of solutions, when it must be immersed in the fluid. In this case it is advisable, in order to avoid overheating, to surround the bulb with asbestos (Beckmann). Regular boiling

is ensured by pieces of platinum foil in the liquid, but only for a time; a platinum wire melted into the bottom of the flask is more effectual (Beckmann, *Zeitsch. f. Phys. Chem.* iv. p. 532, 1889).

[Where pumice stone can be used, a little, rather coarsely powdered, is very effectual in producing regularity in the boiling (*Tr.*)]

27A.—Vapour Tension.

The vapour tension of a body is given by the height of the column of mercury (strictly speaking reduced to 0°; see **20**, 1) which balances the tension.

A Torricellian vacuum is prepared by almost filling with dry mercury (**19**) a glass tube of about a meter long and 10 mm. wide, sealed at one end, sweeping out the air which adheres to the glass by a large air bubble run up and down the tube, or more completely by boiling, and then inverting the tube thus completely filled and closed by the finger into mercury. The height H of the column of mercury is read off on a mm. scale behind the tube, or engraved on it, or with a cathetometer (**18A**). This must be nearly the height of the barometer at the time. Into the vacuum is introduced the substance, free from air, by a syringe in the case of a fluid, inclining the tube till the mercury completely fills it in the case of very volatile substances, otherwise the tube may easily be broken. It is more convenient and safer if the tube has at the upper end a narrow neck, widening out again into a small funnel, with a stopper ground into the neck. The stopper may very well be a thermometer, and is made air-tight by pouring a little mercury into the funnel on to the joint. On this the fluid to be investigated is poured, the stopper cautiously lifted until the mercury and a portion of the fluid have entered, and then again a little mercury poured in.

The height H' of the column of mercury is again read. $H - H'$ is the vapour tension of the substance.

If any alteration of the height of the barometer has taken place the same alteration must be made in H. If the excess of fluid above the mercury has the height h, we must add $h \cdot s/13{\cdot}6$ to H' where s is the specific gravity of the fluid. At higher temperatures the vapour of the mercury itself has a sensible tension, which may be found from Table 14, and

must be added to H'. The readings are all made at the top of the mercury meniscus. Since the surface tension of the mercury is altered by the drop of fluid, and as this in like manner possesses a surface tension, accurate measurements require a wide tube (15 to 20 mm. See **20**, 3).

The smaller the "vacuum" the greater are the errors arising from any residue of air. The stopper may in case of need be rendered air-tight by means of some lubricating material which is not acted upon by the fluid, *e.g.* tallow or vaseline. Before reading off, a fresh portion of the fluid should be brought to the surface by inclining the tube or plunging it deeper into the mercury vessel.

Fluids with high vapour tensions may be examined in the closed limb of a vertical siphon tube. The vapour tension is in this case equal to the atmospheric pressure $\pm$ the difference in height of the columns of mercury in the two limbs of the tube. If the observation is made at higher temperatures the height of the mercury must be corrected for this.

The boiling-point of a fluid (**27**) is that temperature at which the tension of the vapour is equal to the atmospheric pressure.

Vapour Tension of Solutions (Babo, Wüllner, Raoult).—If a not volatile substance, *e.g.* a salt, is dissolved in a fluid which by itself has the vapour tension e, the tension will be diminished by the quantity $\epsilon = e \,.\, \mu/(\mu + \mu')$ where μ and μ' signify the gram-molecules in a volume of the solution of dissolved substance and solvent respectively. That is to say, if the solution contains the quantity p of the substance and p' of the solvent, and the corresponding molecular weights are M and M', we have $\mu = p/M$ and $\mu' = p'/M'$. The lowering of the vapour tension can therefore be used for the determination of molecular weights according to the following rule:—Let p parts by weight of a substance be dissolved in p' parts of a fluid, let the vapour tension of the fluid be e (Table 13, 14), and that of the solution be smaller by the amount ϵ. Then the molecular weights are in the proportion

$$\frac{M}{M'} = \frac{p}{p'} \,.\, \frac{e - \epsilon}{\epsilon}$$

For concentrated solutions variations arise, and for electrolytes the remarks on p. 103 hold good.

Method by Boiling.—The alteration of the vapour tension of a solution can be determined as follows:—A short siphon barometer tube of at least 10 mm. diameter (and better 15 mm.) is filled with mercury, a sufficient quantity of the solution is introduced into the tube, and the whole hung in a chamber heated to the boiling-point of the solvent by a rapid current of its vapour. The difference of level of the mercury in the two limbs of the tube gives the alteration of the vapour tension of the solution. For corrections, see p. 105.

Example.—A solution of $p = 20$ grms. of cane sugar, in $p' = 100$ grms. of water. The difference of level of the mercury amounted to 7·5 mm., and in the closed tube a column 11 mm. high of the sugar solution stood over the mercury. The sp. gr. of the 17 per cent solution at 100° being taken as 1, there must be added to the 7·5 mm. $11/13{\cdot}4 = 0{\cdot}8$ mm.; ϵ therefore is 8·3. The height of the barometer e was 747 mm., therefore

$$M = M' \frac{p}{p'} \frac{e - \epsilon}{\epsilon} = 18 \frac{20}{100} \frac{747 - 8{\cdot}3}{8{\cdot}3} = 320.$$

Actually $$C_{12}H_{22}O_{11} = 342.$$

On the observation of lowering of vapour tensions, Beckmann, *Zeitsch. für Phys. Chemie*, iv. p. 532, 1889.

28.—Determination of the Humidity of the Air (Hygrometry).

The magnitudes to be here determined are—

(1.) The density of the vapour of water in the air—*i.e.* the weight in grams of the water contained in 1 c.c. of air. Since this number is very small, it is usual to multiply it by 1000000, by which we obtain the weight of the water in 1 cubic meter of air, expressed in grams. This is called in meteorology the *absolute humidity* of the air. We shall in the rest of this article call it f.

(2.) The *relative humidity*, degree of saturation, or the ratio of the amount of water actually existing in the air, to the amount which would saturate it. This quantity is

obtained from the absolute humidity f and the temperature of the air, for which latter the maximum amount of vapour f_0 is taken from Table 13, by calculating it as f/f_0.

(3.) The *tension* e of the water vapour in the air, measured in mm. of mercury, which depends on the absolute humidity f, and the temperature t, according to the formula—

$$e = 0{\cdot}943\ (1 + 0{\cdot}00367t) \, . \, f,$$

or

$$f = 1{\cdot}060 \, . \, \frac{e}{1 + 0{\cdot}00367t}$$

so that the determination of t, and either e or f, suffices for the calculation of all the quantities. (e and f have nearly the same numerical value.)

For the vapour density of water is 0·623; therefore 1 cubic centimeter of water vapour of the tension e, at the temperature t, weighs, since it follows, at ordinary temperatures, Mariotte's and Gay-Lussac's law (**16**),

$$0{\cdot}623 \cdot \frac{1293}{1 + 0{\cdot}00367t} \cdot \frac{e}{760} = \frac{1{\cdot}060e}{1 + 0{\cdot}00367t} \text{ grm.}$$

It is convenient to remember that e in mm. and f in grams per cubic meter are, for temperatures from 6° to 30°, when the air is saturated nearly equal to the temperature expressed in centigrade degrees.

I. *Dew-point Hygrometer* (Daniell, Regnault).—With these instruments the dew-point—*i.e.* the temperature τ, at which the air is saturated with vapour—is determined directly. In Table 13 we then find the corresponding quantity of vapour f in a cubic meter of air, or the density multiplied by 1000000, and also the tension e of the vapour saturated at τ; and this is, without any further calculation, the actual tension in the atmosphere. The density needs a correction, because the air in the neighbourhood of the instrument is cooled, and therefore made denser. The contained water as taken from the table for τ is therefore too great, and must, since the vapour expands practically like a permanent gas, be multiplied by $\frac{1 + 0{\cdot}00367 \, . \, \tau}{1 + 0{\cdot}00367 \, . \, t} = \frac{273 + \tau}{273 + t}$, where t signifies the temperature of the air.

The instrument is so arranged that the polished surface reflects to the eye either the light of the sky or a flame. The temperature of the polished surface is then made to sink by the evaporation of ether until a dimness, due to the water condensed on the metal, makes its appearance. Then the evaporation of the ether is interrupted, the temperature rises, and the reading is taken at which the deposit begins to disappear. After some preliminary trials it is easy to bring the temperatures of the appearance and disappearance of the dew within a small fraction of a degree of each other. The mean of the two is then the dew-point τ of the air. If such a regulation of the flow of water from the aspirator of Regnault's hygrometer can be arranged that the deposit of dew sometimes appears and sometimes disappears, the temperature as read off is the dew-point without any further trouble. Care should be taken that the moisture arising from the body, breath, etc., is kept as far as possible from the surface on which the dew is to be formed.

II. *Auguste's Psychrometer* [in England usually called Leslie's].—The humidity of the air is determined from the rapidity with which water evaporates in the air, which rapidity, again, is measured by the cooling of a thermometer the bulb of which is kept wet.

If then

t = the temperature of the air (dry bulb reading);
t' = the wet bulb reading;
e' = the maximum tension of water vapour at the temperature t', as taken from Table 13;
b = the height of the barometer in mm.;

the actual tension of the vapour e is obtained by the formula—

$$e = e' - 0{\cdot}00080 \;.\; b \;.\; (t - t')$$

Or when t' is below freezing-point $e = e' - 0{\cdot}00069 \,.\, b \,.\, (t - t')$. From e the absolute humidity f may be calculated by the formula on the previous page.

The above constants are for observations in the open air in moderate motion. In still air a larger number must be used; that for a small closed room may be as much as

0·0012. Observations in a room are best arranged to fulfil the conditions of the constant 0·00080 by moving the thermometer about (swinging like a pendulum).

On account of the many sources of error to which this method of determining e is subject, it is quite sufficient to use for b a mean height of the barometer. If we take $b = 750$, $e = e' - 0{\cdot}6\ (t - t')$, or, under freezing-point, $0{\cdot}52\ (t - t')$. The value of f may be approximately found by the formula $f = f' - 0{\cdot}64\ (t - t')$, taking from Table 13 the value of f' corresponding to t'. If psychrometer determinations be made, in a moderately large, closed room, the tension e will be found with sufficient accuracy as $e = e' - 0{\cdot}8\ (t - t')$.

Example.—It has been found that $t = 19^\circ{\cdot}50$, $t' = 13^\circ{\cdot}42$, the height of the barometer $b = 739$ mm. We find in Table 13 for t', $e' = 11{\cdot}44$ mm. From this we must take $0{\cdot}00080 \times 739 \times 6{\cdot}08 = 3{\cdot}59$ mm.; therefore the tension of the water vapour $e = 7{\cdot}85$ mm. From this the water contained in 1 cubic meter at the temperature $19^\circ{\cdot}5$ is found, according to p. 108, to be—

$$f = \frac{1{\cdot}060\ .\ 7{\cdot}85}{1 + 0{\cdot}00367\ .\ 19{\cdot}5} = 7{\cdot}8\frac{g}{cb.m.}$$

The relative humidity is $7{\cdot}8/16{\cdot}7 = 0{\cdot}47$.

Regnault's accurate formula $e = e' - \dfrac{0{\cdot}480b\ (t - t')}{610 - t'}$ (or below freezing-point 689 instead of 610) gives practically the same results, except at very high temperatures, as our expression.

III. *Absorption Hygrometer.* — The water contained in 1 cubic meter of air may be obtained directly by drawing a measured quantity of air through a tube filled with calcium chloride, concentrated sulphuric acid on pumice, or anhydrous phosphoric acid, by means of an aspirator, and determining the increase of weight due to the absorption of the water.

IV. *Hygroscopic Bodies.*—The form (curvature, length, twist) of a hygroscopic body depends on the moisture of the air. The scale which generally gives the relative humidity in percentages must be graduated empirically. From time to time the saturation point of the instrument must be verified in a chamber completely saturated with vapour of water.

29.—Determination of Specific Heat. Method by Mixture. Water Calorimeter.

Unit of Heat or Calorie.—This is usually taken as the quantity of heat which warms unit mass (1 grm. or 1 kilo.; gram or kilogram-calorie) of water by 1°. But this quantity itself depends upon the temperature. For a long time the heating of water from 0° to 1° has been taken as the standard according to the procedure of Regnault in his pioneer work on the subject. If now, as is usual, measurements are made with water at the temperature of the room, the numbers found must be reduced to the capacity for heat of water between 0° and 1°. But the ratio deduced by Regnault for this purpose has been made quite doubtful through later experiments.

Other propositions have therefore been made and partly carried out, and there are used at present—

(1.) The water calorie from 0° to 1°.

(2.) The method of mixture compares the capacity for heat of the object with that of water at the temperature of the room. From this point of view the water calorie from 15° to 16° or thereabouts is taken.

(3.) In that method which is the most accurate of those at present employed, that namely by melting ice, it is convenient to make the comparison with water from 0° to 100°. Under the name "Mean Calorie" the one-hundredth part of this quantity of heat referred to the unit of mass of water is preferred by authorities in this field of investigation.

(4.) A further step can be taken by abandoning the heating of water and introducing as the standard the quantity of heat required to melt the unit of mass of ice under the term of Ice-calorie. This is = 79·9 mean calories.

(5.) The use of the ice-calorimeter is freed from all assumptions, if as unit of heat a quantity be taken such that by using it to melt ice 1 grm. of mercury at 0° will enter the calorimeter on account of the diminution of volume of the water. This calorie contains 64·8 mean water-gram-calories **(31)**.

(6.) From the scientific standpoint that amount of heat

should be taken as the unit which is equivalent to the unit of work. The unit of work is that which at a place where the acceleration by gravity = 1 cm/sec^2 would raise 1 gram through 1 cm. The absolute mechanical calorie would be nearly equal to the $\frac{1}{42200000}$ part of the mean water-gram-calorie (see Appendix 7).

Of these calories 1, 2, and 3 are the practical ones. Unfortunately at the present time nothing certain can be said as to their relation to each other. From the specific heat of water for various temperatures given below * referred to that at 0° as unit, we should conclude that the specific heat of water slightly decreases from 0° to 20° then gradually rises up to 100°. Accordingly we may provisionally assume the water-calorie at 0° (No. 1) and the water-calorie at the temperature of the room (No. 2) as nearly equal; if any distinction is to be made the latter would perhaps be $\frac{1}{4}$ per cent smaller. The mean water-calorie (0° to 100°) is apparently larger perhaps by 1 per cent.

The specific heat, c, of a body is the amount of heat or number of calories which heats the unit of mass (grm. or kilo. according to the definition of the calorie) by 1°. Since the capacity for heat of the body is not quite constant, but usually increases more or less with the temperature, it must be stated for what temperature the number given is true. By the method of mixture what is usually measured is the amount of heat given off between 100° and 20°, therefore c will be the

* The following are the results given by eight observers. The observations of Baumgartner and Münchhausen have been calculated by Pfaundler and Wüllner. From Regnault's observations Bosscha has deduced the altered numbers. Dieterici's tables depend on Rowland's determinations of the mechanical equivalent of heat. We have no detailed account of Bartoli and Stracciati's observations. The specific heat of water is given as—

Observer	At 0°	10°	20°	30°	40°	60°	80°	100°	
Regnault .	1·0000	1·0005	1·0012	1·0020	1·003	1·006	1·009	1·013	Pogg. Ann. lxxix. 254
Bosscha .	1·0000	1·0022	1·0044	1·0066	1·009	1·013	1·018	1·022	,, ,, Jub. 549
Baumgartner	1·0000	1·0031	1·0062	1·0092	..	..	..	1·031	Wied. Ann. viii. 652
Henrichsen .	1·0000	1·0036	1·0079	1·0131	1·019	1·033	1·051	1·072	,, ,, viii. 91
Münchhausen	(1·0000)	..	1·0085	1·0127	1·017	1·025	..	..	,, ,, x. 289
Velten . .	1·0000	0·9876	0·9794	0·9746	0·973	0·975	0·980	0·985	,, ,, xxi. 47
Dieterici . .	1·0000	0·9943	0·9893	0·9872	0·993	1·006	1·018	1·031	,, ,, xxxiii. 441
Bartoli and Stracciati	1·0000	0·9949	0·9929	0·9953	..	..	..	..	Beibl. 1891, 762

mean specific heat between 20° and 100°, which will be approximately that at 60°.

The product of the specific heat and the atomic (or molecular) weight of a body is called its atomic (or molecular) heat. The atomic heat of the solid elements is about 6·3, with considerable divergence in some cases, for example, of C, B, and Si.

I. Solids.

The body to be examined is weighed, heated to a measured temperature T, and placed in a weighed quantity of water of temperature t. Let τ be the common final temperature of body and water.

Then if

M be the weight of the body,

m the weight of the water increased by the water equivalent of the rest of the calorimeter (see below),

the specific heat C of the body between τ and T is given by the formula

$$C = \frac{m}{M} \cdot \frac{\tau - t}{T - \tau}$$

For $m(\tau - t)$ is the amount of heat which the water receives; $CM(T - \tau)$ that which is given up by the body, and these quantities are identical.

The heating of the body is performed in a vessel heated from the outside by boiling water or by the steam from boiling water and carefully protected from currents of air through it (Regnault, Neumann, Pfaundler) and must be continued until the thermometer in the enclosure is stationary. During the observation in the calorimeter the water is kept in motion with a small stirrer. It is advantageous to keep the calorimeter covered, since evaporation would cause an error.

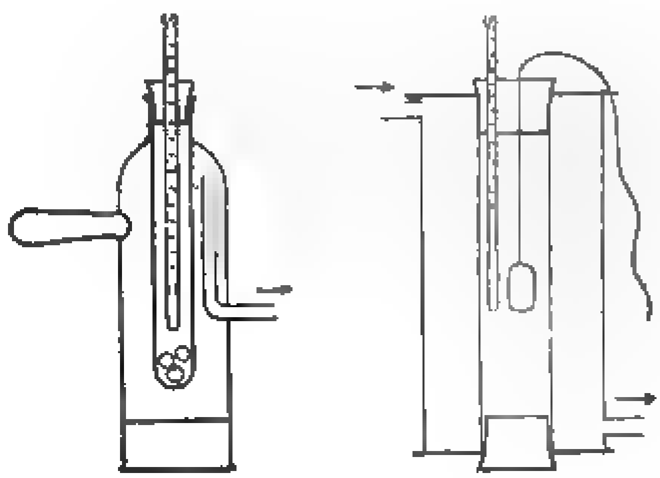


Fig. 14.

If water cannot be used some other fluid (*e.g.* turpentine,

aniline, toluol) of known specific heat (Table 16) must be taken, and the weight of the fluid used multiplied by this number.

Water Equivalent.—It must be noticed that the walls of the vessel and the thermometer participate in the warming. The vessel is made of thin sheet metal (*e.g.* brass or thin silver). If γ be the specific heat of the metal employed (Table 16), μ the weight of the vessel, the quantity of heat necessary to heat it from t to τ will be $\mu\gamma\ (\tau - t)$. The quantity of heat $\mu\gamma$, which raises the temperature of a body 1°, is called its *water equivalent.* The equivalent weight of the thermometer must be determined by experiment. For this purpose it is heated, say by plunging it into heated mercury or over a flame, about 30°, and then quickly transferred to a weighed quantity of water, and the rise of temperature produced is observed. This multiplied by the mass of the water, divided by the loss of temperature of the heated thermometer, gives its equivalent weight.

It will often be sufficient to calculate the water equivalent of the thermometer from the volume v c.c. of the immersed part of the thermometer as $0{\cdot}46\ v$ (Pfaundler). 1 c.c. of mercury has the water equivalent (Tables 1 and 16) $13{\cdot}6 \times 0{\cdot}034 = 0{\cdot}46$ and 1 c.c. of glass has by chance nearly the same, viz. $2{\cdot}5 \times 0{\cdot}19 = 0{\cdot}47$. We may determine v by plunging the thermometer into a calibrated tube.

For m in the above formula must be put the sum of the water equivalents of the solid parts of the calorimeter thus determined once for all, added to the net weight of the water used for filling the instrument.

Loss of Heat.—The vessel for the water is constructed of thin polished metal to diminish the radiation, and is placed on a badly conducting support (3 corks or crossed silk threads). The unavoidable interchange of heat between the calorimeter and its surroundings is, according to Rumford, eliminated by making the initial temperature t of the calorimeter about as much below that of the room as the final temperature τ is to be above it. For this purpose the rise of temperature which may be expected is approximately determined by a preliminary experiment, or where the specific

heat is to some degree known it may be calculated. In order, in addition, that this device may at least approximately suffice, the rise of temperature in the calorimeter should not exceed a moderate quantity (5° or at most 8°). The time also, which is necessary for the transference of the heat from the body to the water should be small, on which account the substance, especially if it is a bad conductor, is used in small pieces, which are either placed in a little basket of wire gauze or threaded on a wire. The water equivalent of the basket is included in the calculations in a manner easily seen.

A method more free from objection is the following:—Let the initial temperature t of the calorimeter be so low that the final temperature τ still remains a little below that of the surroundings. Thus, therefore, the observation is always made with a rising thermometer which is more reliable. A further advantage arises from the fact that the observation of the final temperature τ can be longer continued until the observer is sure that no more excess of heat remains in the body. The whole procedure is then as follows:—

The thermometer is observed say every minute for from 5 to 10 minutes before the heated body is introduced, and the rise of temperature per minute and the excess of temperature of the surroundings thus determined. The body is put in at a noted time and the rising temperature now observed every 20 seconds. From this the temperature correction is easily calculated (see the example below). During the whole time the water is regularly stirred.

If the calorimeter is open, some heat is lost by evaporation, and in such a case the process is completed by an observation of the course of the temperature after the heating. For detailed directions as to the improvement of the results by avoiding loss of heat see *e.g.* Müller-Pfaundler, *Physik.* ii. p. 297; Wüllner, *Exp. Physik.* iii. 4th ed. p. 438.

Example.—(1.) Water equivalent of the vessel and stirrer.—Both parts were made of brass and weighed together, $\mu = 19$ grams The specific heat of brass is $\gamma = 0{\cdot}094$; the equivalent therefore is $\mu\gamma = 19 \times 0{\cdot}094 = 1{\cdot}8$ g.

(2.) Equivalent of the thermometer.—The thermometer was

warmed to 45°, and plunged into a small vessel containing 20 grams of water of the temperature of 16°·25. The temperature then rose to 17°·10. The equivalent of the thermometer therefore amounts to—

$$20 \cdot \frac{17{\cdot}10 - 16{\cdot}25}{45 - 17{\cdot}1} = 0{\cdot}6 \text{ grm.}$$

(3.) The body weighed $M = 48{\cdot}3$ grams.
The water weighed 74·0 g.;
therefore $m = 74{\cdot}0 + 1{\cdot}8 + 0{\cdot}6 = 76{\cdot}4$ „
The temperature of the hot body $T = 99°{\cdot}7$ „
The initial temperature of the water $t = 12°{\cdot}05$ „
The final temperature $\tau = 17°{\cdot}46$ „
Hence we find the specific heat—

$$C = \frac{76{\cdot}4}{48{\cdot}3} \cdot \frac{17{\cdot}46 - 12{\cdot}05}{99{\cdot}7 - 17{\cdot}46} = 0{\cdot}1041$$

(4.) Correction on account of interchange of heat—
The temperature of the surroundings $= 18°{\cdot}0$.

Times	25 m.	26 m.	27 m.	28 m.	29 m.	30 m.
Thermometer	11·54	11·65	11·75	11·88	11·96	12·05

At 30 min. the hot body was introduced and the following readings were then obtained :—

Times	30m.	*20s.*	*40s.*	31m.	*20s.*	*40s.*	32m.	*20s.*	*40s.*	33m.
Thermr.	12·05	14·7	15·9	16·8	17·2	17·3	17·4	17·44	17·45	17·46

In the first set of observations the mean temperature 11°·8 was 6°·2 below the temperature of the surroundings. On this account the thermometer rose in 5 min. $12{\cdot}05 - 11{\cdot}54 = 0{\cdot}51$. Consequently the gain of temperature per degree of excess in 1 minute $= 0{\cdot}5/(5 \times 6{\cdot}2) = 0°{\cdot}0164$.

Therefore in the		1st	2nd	3rd minute
the mean temperature		= 14·9	17·2	17·4
was below that of the surroundings	$\Theta =$	3·1	0·8	0·6
The rise of temperature	$= 0{\cdot}0164 \, . \, \Theta =$	0·051	0·013	0·010 altogether 0°·07.

The observed $\tau = 17{\cdot}46$ must therefore be corrected by $-0{\cdot}07$ and gives the true $\tau = 17{\cdot}39$, and from the previous formula the true $C = 0{\cdot}1027$.

II. *Liquids.*

(1.) The specific heat of a liquid may be determined exactly as above described, if it be enclosed in a vessel, heated in it, and with it plunged into a water-calorimeter. The water equivalent of the vessel is brought into the calculation quite simply.

(2.) If a sufficient quantity of the fluid is available the calorimeter is filled with it, and a weighed body of known specific heat is heated and plunged into it as above described. The body must be a good conductor,—for instance, a wire basket with fragments of glass or copper.

If M, T, C be the weight, temperature, and specific heat of the heated body;

t = the initial temperature of the fluid;
τ = the final temperature;
m = the weight of the fluid;
w = the equivalent of the solid parts of the calorimeter;

the mean specific heat between t and τ of the fluid is—

$$c = C\frac{M}{m} \cdot \frac{T-\tau}{\tau - t} - \frac{w}{m}$$

It is convenient to use as the heated body a glass globe containing about 180 grams of mercury, and provided with a narrow tube marked in two places corresponding with temperatures of about 80° and 25°. This is heated in a mercury-bath, or cautiously over a flame, until the mercury in the apparatus is above the higher mark. It is then allowed to cool, and at the moment when the mark is reached is plunged into the liquid. The liquid is kept stirred, and when the mercury reaches the lower mark the globe is removed and the temperature again observed (Andrews; Pfaundler).

Let m, w, t, τ have the same signification as before; and if a parallel experiment performed with the same heated body, and with a quantity m' of water in the same vessel, gave the initial and final temperatures t' and τ', we have at once

$$c = \frac{1}{m}\left[(m'+w)\frac{\tau'-t'}{\tau - t} - w\right]$$

for

$$(cm + w)(\tau - t) = (m' + w)(\tau' - t')$$

29A.—SPECIFIC HEAT. GALVANIC METHOD (Pfaundler).

Two fluids contained in similar vessels are warmed by the same current of electricity (**63**) which traverses the same resistance of platinum wire, or, better, one of an alloy of platinum and silver in each fluid. The two quantities are suitably so chosen that the rise of temperature to be expected is about the same in each case. The initial temperature is then made as much below the temperature of the room as the final temperature will be above it. By this means the loss of heat during the experiment and the alteration of the resistance of the wire by temperature are in some degree eliminated.

The quantity m of the fluid together with the water equivalent w of its vessel and thermometer is heated from t to τ; the other quantity m', with the water equivalent w' of its vessel and thermometer from t' to τ'.

Then

$$\frac{cm + w}{c'm' + w'} = \frac{\tau' - t'}{\tau - t}$$

If the fluid m' is water, we have

$$c = \frac{1}{m}\left[(m' + w')\frac{\tau' - t'}{\tau - t} - w\right]$$

Possible irregularities in the relative conditions are most simply eliminated by reversal of the fluids and taking the mean of the two results found.

Errors may arise from possible differences in the temperatures, and therefore the resistances, of the wires, produced by unequally rapid cooling, and from the possibility that part of the current may be conducted through the fluid instead of through the wire. Pure water conducts very badly; a leaking of the current need not be feared if the tension in the wire is under 2 volts (**63**, I.) The resistance should not be made too great. In fluids which conduct the current, glass spirals filled with mercury may be used. The ratio of the resistances R/R' of the two wires may be obtained during the experiment by shunting a portion of the current (**71**, III. and **71**A), or by

arranging the wires as arms of a Wheatstone's bridge (**71B**). $(\tau' - t')/(\tau - t)$ must then be multiplied by R/R'.

(See Müller-Pfaundler, *Lehrbuch der Physik.*, 8th ed. ii. 2, p. 311.)

30.—Specific Heat. Method by Cooling. (Dulong and Petit.)

Here the times are compared in which heated bodies, which cool under the same conditions, experience the same fall of temperature. The process only furnishes useful results in the case of liquids or solids of good conducting power.

A small vessel of thin polished metal, in which a thermometer is placed, is filled with the substance. Solid bodies may be powdered and tightly rammed down. It is then warmed with the substance in it, and introduced into a metal receiver, which can be exhausted of air, and the temperature and the time are observed. The receiver is kept at a constant temperature by surrounding it with a large quantity of water or with melting ice.

For quantities of liquids not too small the rate of cooling in the air in one and the same closed metallic vessel may be observed.

Let there be two sets of observations with the vessel filled with two different substances. We will call

m and M the quantities used to fill the vessel;
w the water equivalent of the vessel and thermometer (p. 114);
z and Z the times during which the bodies cool from the same initial to the same final temperature;
c and C the two specific heats;

then—

$$c = \frac{1}{m}\left[(MC + w)\frac{z}{Z} - w\right]$$

For the times necessary to the same amount of cooling are proportional to the quantities of heat given off—*i.e.*

$$\frac{z}{Z} = \frac{mc + w}{MC + w}$$

If, therefore, we know C; *e.g.* by using water, $C = 1$, we can from this find c.

Errors of observation have the least influence when the excess of the first temperature over that of the surroundings is two to three times that of the second.

If the temperature of the surrounding walls of the receiver be not the same at the two experiments, the temperature of the substance must be taken as the excess over that of the receiver.

Some time must be allowed to elapse after warming the body before commencing to observe. It will always be best to make a set of observations by noticing the temperature say every 30 seconds. Then a curve is constructed from these observations by putting the times as abscissæ, the temperature as ordinates, and from the curves are taken the times which correspond to equal initial and final temperatures (or excess of temperature over that of the surrounding bodies). Thus we can, from one pair of observations, obtain a large number of determinations, of which the mean is afterwards taken (see also **3**, III.)

The two experiments may also be performed at the same time in two vessels as nearly similar as possible. The experiment is repeated with the fluids changed and the mean of the times taken for each fluid. By this means any want of similarity in the vessels is eliminated.

31.—Specific Heat. Ice-Calorimeter.

Old Method (Lavoisier and Laplace).—The body, of weight m, heated to the temperature $t°$, is placed in dry ice at 0°. If, by this means, the quantity M of ice be melted, the specific heat of the body is measured in mean water calories, p. 111.

$$c = \frac{M}{m} \cdot \frac{79{\cdot}9}{t}$$

The unit-weight of ice at 0° requires 79·9 calories to become converted into water at 0°.

The access of heat to the ice-calorimeter from the outside is avoided by surrounding it on all sides with melting ice.

In order to determine the quantity of ice melted by weighing, or taking the volume of the water, with anything approaching to accuracy, we must, on account of the adhesion of the water to the ice, use a large quantity of the body.

For approximate determination, we may employ a piece of ice with a smooth surface, with a hollow in which the heated body is placed. During cooling, this is enclosed with a smooth cover of ice. Afterwards the melted water is absorbed with a cold bit of sponge and weighed in it (Black).

Bunsen's Ice-Calorimeter (*Pogg. Ann.* vol. cxli. p. 1; [*Phil. Mag.* 1871]).—In this form of instrument the quantity of ice melted is determined by the diminution of volume which is experienced when water passes from the solid to the liquid state. If a mixture of ice and water contract v c.c., whilst a body of the mass of m g. cools from $t°$ to 0°, the specific heat of the body is—

$$c = \frac{v}{m} \cdot \frac{881}{t}$$

1 g. of ice has, according to Bunsen, the volume 1·0908 c.c., whilst 1 g. of water at 0° has the volume 1·0001 c.c. By melting 1 g. of ice, which requires 79·9 units of heat, there occurs therefore the diminution of volume of 0·0907 c.c. The unit of heat therefore diminishes the volume $\frac{0 \cdot 0907}{79 \cdot 9} = \frac{1}{881}$ c.c.

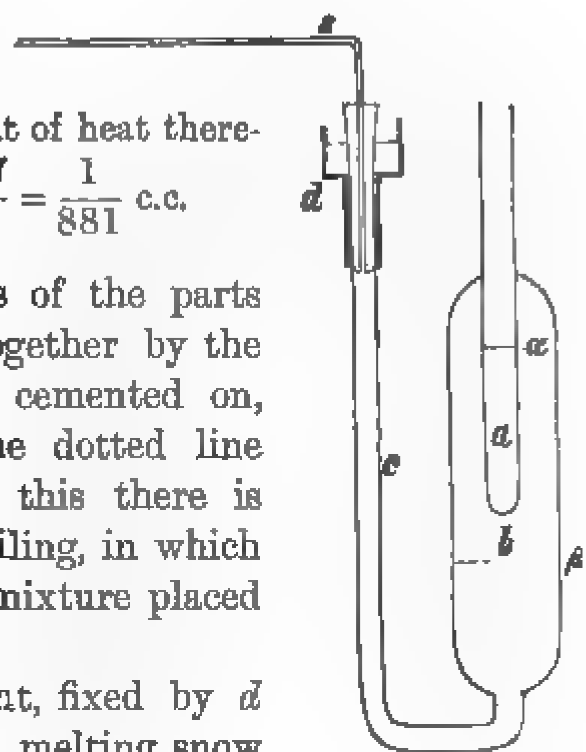


Fig. 15.

Bunsen's calorimeter consists of the parts a b c, made of glass, sealed together by the blowpipe; d is an iron piece cemented on, b c and d are filled up to the dotted line with boiled mercury. Above this there is in b water freed from air by boiling, in which the ice is formed by a freezing mixture placed in a.

When in use the instrument, fixed by d in a holder, is surrounded with melting snow or pure ice, and the calibrated scale-tube s pressed through a long cork fixed in d until the mercury stands sufficiently far along the divisions. When the vessel

a has been filled up to a with water or some other fluid which does not dissolve the body to be experimented on, this latter is heated, and let fall into a, which contains a little of some soft substance to prevent breakage, and a cork is then inserted. The mercury in s sinks, and finally becomes stationary. If the movement of the mercury amount to e scale-divisions, and if the volume of 1 division be A, $v = Ae$.

Calibration of the Tube.—We get A by determining the weight, μ grams, of a thread of mercury which occupied n divisions. If τ be the temperature at the time when this measurement is made—

$$A = \frac{\mu(1 + 0{\cdot}00018\tau)}{13{\cdot}596n} \text{ c.c.} \quad \textbf{(19)}$$

If now in the equation for c, Ae is substituted for v and for $13{\cdot}596/881$ we write $0{\cdot}01544$, the value in heat units (grm.-calories) K of 1 scale-division is—

$$K = \frac{\mu}{n}\,\frac{1 + 0{\cdot}00018\tau}{0{\cdot}01544} \text{ and then simply } c = Ke/mt$$

Empirical determination of K.—A light glass bulb (0·5 to 1 c.c.) is filled, leaving a little space for expansion, with water, weighted with some platinum, heated to the temperature t (p. 113) and introduced into a. If w is the sum of the water equivalents, e' the scale movement produced, then

$$K = wt/e'$$

Method by weighing.—Instead of reading off the thread of mercury in the tube, this suitably bent and quite full is plunged into a vessel containing mercury and the amount of the metal sucked up on the introduction of the warm body determined by the difference of the weights of the vessel. 0·01544 grm. of mercury corresponds to the gram-calorie.

Slight impurities of the snow or ice with which the calorimeter is surrounded are sufficient gradually to alter the position of the thread of mercury. The rate of change must be observed, and taken into account for the time of the experiment.

Or the freezing-point of the water is so far depressed by increasing the pressure by increasing the column of mercury until the movement of the end ceases (Dieterici). The capillary tube is for this purpose bent twice, so that the horizontal part with the scale for the reading, or the opening into the vessel, can be placed higher or lower.

(*Cf.* Bunsen, *loc. cit.*; Dieterici, *Wied. Ann.* xxxiii. 418, 1888, xxxviii. 1, 1889; Schuller and Wartha, *ib.* ii. 359, 1877; where also the determination of the heat of chemical combination is treated.)

31A.—Specific Heat. Vapour Calorimeter (Joly; Bunsen).

The body m is suspended from a balance by a fine wire in a space into which steam from boiling water can be suddenly introduced through a wide tube. The amount of water w condensed on the body is weighed. Unit weight corresponds to 536 calories. The specific heat, therefore, if the initial temperature is t_0 and the temperature of the steam T, is (Table 13A)

$$c = \frac{w}{m} \cdot \frac{536}{T - t_0}$$

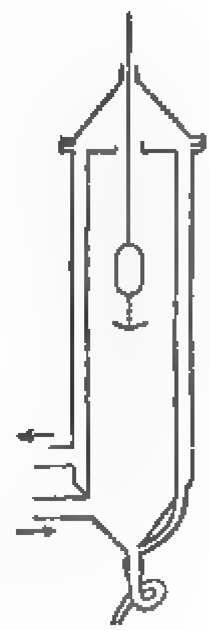


Fig. 16.

The steam escaping round the suspending wire is drawn off by a water pump, or by the draught of a chimney from the opening, which is provided with a plug of plaster having a hole bored through it. To guard against the water dropping from the body, a little saucer of thin platinum-foil is fixed beneath it, the water equivalent being subtracted from mc. (**29**; Table 16). Before weighing, the current of steam is diminished, otherwise the apparent weight is influenced.

The method must be used very carefully, but then appears to furnish very accurate results.

Joly, *Proc. Roy. Soc.* xli. 352, 1886; xlvii. 218, 1889; Bunsen, *Wied. Ann.* xxxi. 1, 1887.

31B.—Thermo-Chemical Measurements.

For measuring the production of heat in chemical reactions, the ice-calorimeter is often suitable, allowing the chemical process to take place on the body previously cooled to 0° in the instrument. A more simple apparatus is the following (Nernst):—Within a wide glass there is supported on corks a beaker holding about a liter. Through a wooden cover are passed a delicate thermometer, a stirrer, and a thin-walled test-tube in which the reaction takes place. If the heat of dilution or solution is to be measured the substance is placed in the test-tube, and when equilibrium has been established in the temperatures the bottom is broken. Only small differences of temperature are employed.

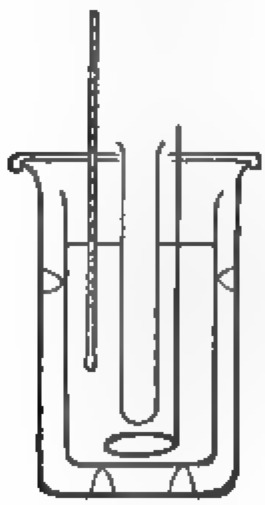


Fig. 17.

In this case the heat developed is calculated as follows (**29**, I.). Let the fluid m in the beaker have the specific heat c, the introduced body m' the specific heat c'; the sum of the water equivalents of beaker, test-tube, stirrer, and thermometer be w (p. 114), and the temperature rise from t to τ, then the heat developed amounts to

$$(cm + c'm' + w)(\tau - t)$$

Precautions and corrections as to interchange of heat enter into the calculations just as on p. 114.

32.—Conducting Power for Heat.

Comparison of the Conducting Power for Heat of Two Rods.—The conducting power for heat or the coefficient of conductivity k is the quantity of heat which passes in unit of time through unit of section, when perpendicular to this section there is unit fall of temperature, *i.e.* when over the unit length the alteration of temperature is 1. By dividing k by the density, multiplied by the specific heat, we obtain what is called the coefficient of temperature conduction. A method first used by Despretz, which, however, requires great precautions to furnish useful results, is as follows:—

We assume that the two rods have the same section, and we give them a similar condition of surface by polishing and electroplating with silver or nickel. The two ends of the rod are brought to different temperatures, say by surrounding one end with boiling water and the other with melting ice. An inferior method is to leave one end exposed to the air, and heat the other by a lamp which burns very regularly. The middle part of the rod, at which the following determinations of temperature are made, is protected by screens from the radiation of the source of heat.

The distribution of temperature, after a time, becomes constant. When this state has been arrived at, the temperatures of three points of the rod equally distant from each other, I. II. III., are measured. The excess of temperature over that of the surrounding air may be called u_1, u_2, u_3.

Let us call

$$\tfrac{1}{2}(u_1 + u_3)/u_2 = n$$

The same course of proceeding is now gone through with the other rod. The excess of temperature at the three points at the same distance from each other as before we call U_1, U_2, U_3, and also—

$$\tfrac{1}{2}(U_1 + U_3)/U_2 = N$$

Then the two conductivities k and K are in the ratio—

$$\frac{K}{k} = \left[\frac{log.(n + \sqrt{n^2 - 1})}{log.(N + \sqrt{N^2 - 1}}\right]^2$$

Proof.—When the thermal condition of the bar has become stationary, each element of length dx of the rod receives by conduction in the unit of time as much heat as it gives off to the surroundings. This last quantity is $a.u.dx$, if a represents the "external conductivity" referred to unit of length of the rod. The former is $k.q.\frac{d^2u}{dx^2}dx$. a and the sectional area q are the same for both rods. Equating the two expressions furnishes the differential equation $\frac{d^2u}{dx^2} = \frac{a}{kq} \cdot u$; of which the complete integral is

$$u = C_1 e^{\sqrt{\frac{a}{kq}} \cdot x} + C_2 e^{-\sqrt{\frac{a}{kq}} \cdot x}$$

Where C_1 and C_2 are two constants depending upon the heating of

the end surfaces. If we call u_1, u_2, u_3, the temperatures for three sections lying at the distance l from each other, we obtain by putting x, $x+l$, and $x+2l$ in the above equation after elimination of C_1 and C_2 the expression—

$$e^{\sqrt{\frac{a}{kq}}\,.\,l} + e^{-\sqrt{\frac{a}{kq}}\,.\,l} = (u_1 + u_3)/u_2 = 2n \text{ (see above)}$$

Hence

$$e^{\sqrt{\frac{a}{kq}}\,.\,l} = n + \sqrt{n^2 - 1} \qquad \text{or } \frac{a}{kq}\,l^2 = [\log.(n + \sqrt{n^2-1})]^2$$

The logarithms are the *natural* logarithms. The equation with K and N substituted in the case of the second rod, and divided into the first, gives the expression which was to be proved.

The temperatures are determined by means of thermo-elements (25), one junction being inserted in small holes in the rod, whilst the other connections are in the surrounding air. It is even sufficient to hang the thermo-element, which may consist of quite fine wires of German-silver and iron, over the rod, having the junction lying on the rod and keeping the wires in place by small weights.

See also Wiedemann and Franz, *Pogg. Ann.* lxxxix. 497, 1853.

Absolute Conductivity.—If the external conductivity a of a rod is known it follows from the last equation in the proof given above

$$k = \frac{a}{q}\,\frac{l^2}{[\log.(n + \sqrt{n^2-1})]^2}$$

A rough determination of a/q can be made as follows:—The rod is uniformly heated, supported as before, and the excess of temperature above that of the air u', u'', . . . observed by means of a thermo-element at several times t', t''. . . . The temperature differences should be of the same order of magnitude as the u_1, u_2, u_3, observed previously. If further the density is s and the specific heat c (Tables 1 and 16) we have

$$\frac{a}{q} = cs\,\frac{\log.u' - \log.u''}{t'' - t'} \text{ (natural logs.)}$$

For if in time dt the temperature alters by du the amount of heat given off by unit length of the rod is on the one hand $= audt$ and on the other $= -qcsdu$. Hence $\frac{du}{u} = -\frac{a}{qcs}dt$ of which the integral is $\log.u = C - \frac{a}{qcs}t$, and therefore $\log.u' - \log.u'' = \frac{a}{qcs}(t'' - t')$.

In the accurate measurement of the absolute conductivity the external conductivity is either eliminated by periodic heating of one end of the rod, or the determination is rendered independent of it by making the observations extremely rapidly after the sudden heating of one side of a body. The problem is one of exceeding difficulty.

See Angström, *Pogg. Ann.* cxiv. 513, 1861, and cxxiii. 628, 1864; Heinrich Weber, *Pogg. Ann.* cxlvi. 257; Kirchhoff and Hansemann, *Wied. Ann.* ix. 1, 1880; F. Weber, *Wied. Ann.* x. 103, 1880; Lorenz, *Wied. Ann.* xiii. 422, 1881.

Cf. Table 10.

ELASTICITY AND SOUND.

33.—Determination of the Modulus of Elasticity by Stretching.

The modulus of elasticity or coefficient of elasticity which denotes the elastic strength of a material is deduced from the elongation which a cylinder (wire or rod) undergoes by means of a stretching force. If q denote the sectional area, l the length of the cylinder, λ the elongation produced by the force P, the modulus of elasticity is

$$E = \frac{l}{\lambda}\frac{P}{q}$$

In other words: the modulus of elasticity is the ratio of the tension exerted on a cylinder of unit length and unit sectional area to the elongation produced by it. Or equally well: the modulus of elasticity is that weight which must be hung on a wire of unit sectional area in order to double its length; provided, of course, that with such stretching the elongation remained proportional to the load.

The magnitude of the number E obviously depends on the units in which the section and the weight are measured.

Ordinary Technical Definition.—It is usual to take the square millimeter and the kilogram as units which is denoted by $kg - wt/mm^2$ placed after the number (Table 17). Strictly speaking the variation of gravity should be taken into account and the observations reduced, for instance, to latitude 45° (p. 77). But the measurements are not usually so exact as to make this correction noticeable.

Modulus of Elasticity in the "Absolute System."—Considering

the gram, kilogram, etc., not, as above, as units of weight, but units of mass, the weight P of a body is $g.P$, where g is the acceleration by gravity. The force 1 would produce a stretching, etc. g times smaller, and the modulus of elasticity appear g times larger than before. A modulus of elasticity, therefore, which is expressed in $kg.\text{-}wt./mm.^2$ units must, in order to reduce it to the "absolute" cm.-grm.-sec. system, be multiplied first by $kg./grm. = 1000$, then by $cm.^2/mm.^2 = 100$, and finally by $g = 981$ $cm./sec.^2$, altogether therefore by 98100000. The number $[E]$ thus obtained signifies, according to Clausius, the number of "dynes" with which a wire 1 square centimeter in section must be stretched in order to double its length, that is, the number of grams which must be hung to it to effect this, but at a place where the acceleration by gravity amounts to 1 $cm./sec.^2$ $[E]$ divided by the density gives the square of the velocity of sound in the material in $(cm./sec.)^2$. (See Appendix 6, also **10A**.)

We shall retain the ordinary technical definition of E.

Determination of the Modulus of Elasticity.—The upper end of the wire or rod under experiment is fixed to the wall or some solid support, and the lower end loaded, when necessary, with a weight sufficient to keep it stretched. An additional weight is now put on to the lower end and the elongation thereby produced is measured. Calling

P, the additional load;
l, the length;
λ, the increase of length caused by P expressed in the same unit as l;
q, the sectional area of the wire in $mm.^2$ see below;

the modulus of elasticity E of the stretching is

$$E = \frac{l}{\lambda} \frac{P}{q} \frac{kg.\text{-}wt.}{mm.^2}$$

If the upper end of a thin wire can be assumed to be perfectly fixed the elongation may be measured by the displacement of a mark on the lower end. Any yielding of the support can sometimes be obviated by applying to it, by means of a cord and pulley, a force in an upward direction about equal to the stretching load applied to the wire. It is usually, how-

ever, better to make a mark on the wire near the upper end and near the lower end, and determine their distance from each other with each load.

For measurements with a microscope movable on a measuring rod (cathetometer), or better with two fixed microscopes provided with eye-piece micrometers, the marks may be fine lines scratched on the wire with a diamond or fine file or suitable marks on pieces of paper cemented to the wire. The greatest elongation used in the measurement must always be within the limits of elasticity; that is to say, the wire must, on the removal of the weight, return to its original length—a condition the fulfilment of which should be verified after the experiment. The limit of elasticity may be widened by loading the wire heavily before the experiment. Even with hard metals the weight employed in the measurements should not exceed half the breaking-strain. (See Table 17 for the tensile strength of some substances.)

On account of elastic "fatigue" the elongations of most materials increase more or less—least with steel—in the course of time. It is usual to allow the load to act for as short a time as possible; the slight alteration of temperature which accompanies the elongation has no noticeable influence on the results. Strictly speaking, two moduli of elasticity should be distinguished, one with brief loads the other with more lasting ones, of which the latter may be less by possibly 2 per cent.

The accuracy of the results will be considerably increased if the length be observed under many loads. (See the example, or, for the calculation by the method of least squares, **3**.)

The law that the elongation is proportional to the load is only an approximation. The elongation λ actually increases a little faster than the load. This can be nearly expressed by

$$\lambda = \frac{El}{q}(P + A \cdot P^2)$$

See J. O. Thompson, *Wied. Ann.* xliv. 555, 1891.

Determination by bending a Stretched Wire.—The elongation of thin wires can be determined by clamping the horizontally stretched wire firmly at the ends and placing a weight in the

middle so that it becomes bent. Let l be the whole length of the wire. Let two different weights P_1 and P_2 produce the fall H_1 and H_2 respectively at the middle point of the wire, to be measured from the line connecting the points of fixing; the modulus of elasticity is then (H_1 and H_2 being small compared with l)

$$E = \frac{1}{8} \cdot \frac{l^3}{q} \cdot \frac{\frac{P_2}{H_2} - \frac{P_1}{H_1}}{H_2^2 - H_1^2}.$$

For greater depressions this result must be multiplied by

$$1 + 3\frac{H_1^2 + H_2^2}{l^2}$$

The two parts of the numerator differ but little, so that H_1 and H_2 must be accurately measured.

The proof is as follows. The elongation of each half of the wire $\frac{1}{2}l$ is plainly $\lambda = \sqrt{(\frac{1}{2}l)^2 + H^2} - \frac{1}{2}l$, or approximately, according to formula 3, p. 10,

$$\lambda = \tfrac{1}{2}l(\sqrt{1 + 4H^2/l^2} - 1) = \tfrac{1}{2}l(1 + 2H^2/l^2 - 1) = H^2/l.$$

Resolving the weight P_1 into two tensions along the two halves of the wire, we have for one of these the value $\frac{P_1}{2}\frac{\sqrt{(\frac{1}{2}l)^2 + H^2}}{H}$ or, when H is small, $P_1 \cdot \frac{l}{4H_1}$. Let the original unknown tension of the wire be P_0. We have therefore

$$P_1\frac{l}{4H_1} - P_0 = \lambda\frac{2}{l}Eq = 2\frac{H_1^2}{l^2}Eq$$

similarly

$$P_2\frac{l}{4H_2} - P_0 = 2\frac{H_2^2}{l^2}Eq.$$

Subtraction eliminates the unknown P_0, and the expression for E given above is obtained.

Measurement of the Sectional Area.—The section of a wire can be determined by measuring its diameter, using for small sizes a contact lever or microscope (**18**). But the area may also be obtained by weighing. If s (**13B** 2, and Table 1) be

the density of the substance, and if, further, h mm. of the wire weigh m mgrm., the sectional area is $q = m/hs$ mm^2.

Example.—Two meters of an iron wire weighed 1310 mgr.; the density was 7·61; the section was therefore $q = 1310/(2000 \cdot 7{\cdot}61) = 0{\cdot}0861$ mm^2.

The following observations were made in the order of the numbers:—

No.	Load.	Length.	No.	Load.	Length.	Elongation by 2 kgr.
1	0·5 kg.	913·80 mm.	2	2·5 kg.	914·91 mm.	1·11 mm.
3	0·6 „	913·86 „	4	2·6 „	914·95 „	1·09 „
5	0·7 „	913·90 „	6	2·7 „	915·00 „	1·10 „
7	0·8 „	913·98 „	8	2·8 „	915·09 „	1·11 „

The elongation for $P = 2{\cdot}00$ kg. is therefore, taking the mean, $\lambda = 1{\cdot}102$ mm. Consequently the modulus of elasticity is (p. 128)

$$E = \frac{l}{\lambda} \cdot \frac{P}{q} = \frac{913{\cdot}8 \,.\, 2}{1{\cdot}102 \,.\, 0{\cdot}0861} = 19260 \ [kg.\text{-}wt./mm.^2]$$

In the absolute cm.-grm. system this modulus is (p. 129)

$$[E] = 19260 \,.\, 98100000 = 1890 \,.\, 10^9 [cm.^{-1}\ g.\ sec.^{-2}]$$

34.—Modulus of Elasticity by Longitudinal Vibrations.

A rod held at the centre, or a wire stretched and held firm at the two ends, is made to give out its fundamental note by rubbing, a rod being rubbed at the free end, a wire in the middle. The wave-length is then equal to twice the length of the rod or wire $2l$. If therefore N be the pitch of the note, *i.e.* the vibrations per second, the velocity of sound in the material is

$$u = 2Nl$$

If l be measured in cm. and s be the density of the substance, the modulus of elasticity in absolute units (p. 129, and App. 10A) will be

$$[E] = u^2 s = 4n^2 l^2 s [cm.^{-1} g.\ sec.^{-2}]$$

$[E]$ divided by 98100000 gives the modulus E in the practical, technical $[kg.\text{-}wt./mm.^2]$ units (p. 128). This comes to the same thing, as is readily seen, as measuring l in meters, and therefore expressing u in m./sec. and then reckoning directly

$$E = \frac{u^2 s}{9810} = \frac{4n^2 l^2 . s}{9810} [kg.\text{-}wt./mm.^2]$$

The longitudinal vibrations are produced by rubbing with a woollen cloth, which for metal or wood is sprinkled with resin, for glass is damped.

The note is determined by comparison with a tuning-fork of known pitch. The estimation of intervals, which is an uncertain matter, may be reduced by the use of a monochord to a comparison of lengths (**37A**, 4).

It is often difficult to determine the particular octave in which the note lies, as the pitch is mostly very high. An error in this will be easily noticed, because it always makes the result at least four times too great or too small. For the determination of the pitch of a note by dust figures see **37**, for graphical determination **37A**.

The modulus of elasticity deduced in this manner may differ somewhat from that found by the elongation method, firstly on account of the warming and cooling by compression and extension, and secondly because between putting on the load and taking the reading of length some time elapses, and during this time a slight elongation may take place through the elastic "fatigue" (p. 130).

Example.—The above-mentioned iron wire, of the length of 1·361 meter, gave the note $A\sharp_3$, which is found by Table 18 to be produced by 1865 vibrations per second. The specific gravity is 7·61; therefore

$$E = \frac{4 \times 1865^2 \times 1{\cdot}361^2 \times 7{\cdot}61}{9810} = 20000 \frac{kg.\text{-}wt.}{sq.\ mm.}$$

35.—Modulus of Elasticity by bending a Rod.

I. *Rod clamped at the end.*—A horizontal rod is clamped tightly at one end, and the position of the free end observed on a vertical scale (*e.g.* on a scale engraved on a mirror close behind it, or by means of a cathetometer). It is then loaded with a weight of P kilogrammes on the free end, and the amount of deflection S thus produced is observed. Let the free length of the rod be l. Then the modulus of elasticity

E, if the section of the rod be a rectangle with the vertical side a and the horizontal b, is

$$E = 4\frac{P}{S} \cdot \frac{l^3}{a^3b}$$

if the section be a circle of radius r, instead of a^3b write $3r^4\pi$.

Thin Wires.—This method of determination is also specially applicable to thin wires. The diameter is obtained from the weight and the specific gravity. Deviations from exactly circular figure are eliminated by making a second experiment with the horizontal and vertical diameters exchanged in position.

II. *Rod supported at both ends.*—The difficulty of getting a perfectly tight clamping is avoided by laying the rod with both ends loose upon two solid supports. Let the distance of the two supports from each other be l. A weight P is then hung from the middle of the rod and produces the deflection $\frac{q}{s}$, best read off on a mirror scale close behind, and we have, for rectangular section (*vide supra*),

$$E = \frac{1}{4} \frac{P}{s} \frac{l^3}{a^3b}$$

Reflection.—It is far more accurate to measure, instead of the depression of the centre, the *inclination* of the ends (Kirchhoff, Pscheidl). Let the load P produce the angle of inclination ϕ in an end surface, then

$$E = \frac{3}{4} \frac{l^2}{a^3b} \frac{P}{\tan \phi}$$

To measure ϕ a little vertical mirror is fixed to the end of the rod and the rotation observed with a telescope and vertical scale (**48, 49**). It is better to observe at both the ends and take the mean. Or instead, two mirrors may be fixed at the two ends, facing each other, but slightly inclined to each other so that the ray of light from the scale is reflected from one to the other and thence to the telescope (A. König, *Weid. Ann.* xxviii. p. 108, 1886). The scale and telescope now of course stand opposite each other. If A be the distance of the scale from

the mirror facing it, and d the distance apart of the mirrors, both measured in scale-divisions, we may write with sufficient accuracy

$$tan\ \phi = \frac{n}{4A + 2d}$$

P is expressed in kg., all lengths in mm., in order to get our result in the ordinary unit of the modulus of elasticity (p. 128).

The formulæ given above assume that the deflections are small compared with the length. We must also make sure that the change of form is within the limit of recovery—*i.e.* that on taking away the weight the original form is resumed. Small sections are determined by weighing (p. 131), and the above formulæ may then be simplified by remembering that ab and $r^2\pi$ are the respective sections of the rods.

If the height a of the rod cannot be neglected in comparison with the length l the value of E calculated from the formula given above must be multiplied by $1 + 3a^2/l^2$.

(*See* Koch, *Wied. Ann.* v. 353, 1878.)

The equation under I. for rectangular rods is got thus :—When the rod is bent the fibres at the top are stretched, those at the bottom compressed, the middle layer remains of unaltered length. We denote by x the horizontal co-ordinate of a point of this "neutral plane" measured from the fixed point, and by y the vertical co-ordinate; and then the curvature of the rod at any point will be $\frac{d^2y}{dx^2}$, for we assume that the bending is small. If now z be the distance of a fibre from the neutral plane (above being reckoned positive, below negative), a small portion of the fibre is stretched or compressed in the ratio $z\frac{d^2y}{dx^2}$ to its original length. A lamina of the breadth b, and thickness dz, seeks therefore to draw itself together with the force $Ez\frac{d^2y}{dx^2}bdz$, and these forces in the laminæ, distant $+z$ and $-z$, produce a couple equal to $2Ez^2\frac{d^2y}{dx^2}bdz$. The couple, therefore, developed in one entire section of height a and breadth b, is—

$$2Eb\frac{d^2y}{dx^2}\int_0^{\frac{a}{2}} z^2dz = Eb\frac{a^3}{12}\frac{d^2y}{dx^2}$$

This couple, produced by the elasticity, must be equal to the statical moment $P(l-x)$, exercised by the weight at the place, therefore—

$$\frac{d^2y}{dx^2} = \frac{12}{E}\frac{P}{a^3b}(l-x)$$

whence

$$\frac{dy}{dx} = \frac{12}{E}\frac{P}{a^3b}\left(lx - \frac{x^2}{2}\right)$$

and

$$y = \frac{12}{E} \cdot \frac{P}{a^3b} \cdot \left(\frac{lx^2}{2} - \frac{x^3}{6}\right)$$

therefore the depression S and deflection *tan* Φ of the end, where $x = l$

$$S = \frac{4}{E} \cdot \frac{Pl^3}{a^3b} \text{ and } \tan\Phi = \frac{6}{E}\frac{Pl^2}{a^3b}$$

whence follows the formula in I.

These expressions relate to a rod fixed at one end. But since a rod when lying loose on both ends may be looked upon as if drawn up at each end by a force equal to $\frac{1}{2}P$ and fixed in the middle, the effective length therefore becoming $\frac{1}{2}l$, the inclination *tan* ϕ will be eight times and the deflection s sixteen times smaller than *tan* Φ and S. Hence the formula under II.

Sections of other Shape.—If the section is considered as a plate which possesses unit mass with unit surface, $\frac{1}{12}a^3b$ is the "moment of inertia of the section" of rectangular form referred to the horizontal line passing through its centre of gravity (**54**). Calling this K we may write

$$E = \frac{1}{3}\frac{1}{S}\frac{Pl^3}{K} \qquad \text{or } \frac{1}{48}\frac{1}{s}\frac{Pl^3}{K}$$

In this form the equations are valid for rods of any form of section if the horizontal axis is a principal axis. For instance, the moment of inertia of a circle is $\frac{1}{4}r^4\pi$, whence the formula given for circular sections result.

(For the influence of structure on determinations of elasticity and the modulus of elasticity of crystalline bodies, see the works of W. Voigt in *Wied. Ann.* and *Gött. Nacht.*)

36.—Modulus of Torsion by Oscillation.

Let there be hung on a wire of radius r and length l a mass of moment of inertia K, referred to the wire as axis of revolution (**54**). Let the time of an oscillation under torsion be t sec. (**52**). Then if r and l be measured in cm., K in grm. cm.^{2}, the modulus of torsion in the absolute system is

$$[F] = 2\pi\frac{Kl}{t^2r^4}\,[cm.^{-1}\,g.\,sec.^{-2}]$$

The practical, technical number is obtained by measuring r and l in mm. and K in kg.-mm.^{2}, also adding the factor $1/g = 1/9810$.

$$F = \frac{2\pi}{9810}\cdot\frac{Kl}{t^2r^4} = 0{\cdot}0006405\,\frac{Kl}{t^2r^4}\,[kg.\text{-}wt.\text{-}mm.^{-2}]$$

If a cylinder with its axis vertical be used as the weight $K = \frac{1}{2}MR^2$ where R is the radius in mm., M the mass in kg.

Again as on p. 129 $[F] = 98100000\ F$.

Explanation.—The modulus of torsion, or second modulus of elasticity F, has the following significance. Imagine a plate of unit area cut out of the substance, in which a line is drawn normal to the principal surface. Let one principal surface be fixed and a force k be applied to the opposite one in its own plane and distributed uniformly over it. By this means the layers of the plate will be displaced with respect to each other, and the line which had been normal will now make a small angle δ with the normal. Then F is the ratio of the force k to this angle, or $k = F\delta$.

To the modulus of elasticity E, obtained by stretching, the second modulus F is related as follows:—The stretching of a rod by a hanging weight is accompanied by a lessening of its diameter, as is known by experience. If l be the length, d the diameter, and δ this diminution, which is produced by the extension λ, and if we take the ratio of the contraction of diameter to the increase of length as $\frac{\delta}{d} : \frac{\lambda}{l} = \mu$, then according to the theory of elasticity $F = \frac{1}{2}\frac{E}{1+\mu}$. By experience $\mu \begin{smallmatrix} >0 \\ <\frac{1}{2} \end{smallmatrix}$, therefore in any case $F \begin{smallmatrix} <\frac{1}{2}E \\ >\frac{1}{3}E \end{smallmatrix}$ For the mean value $\mu = \frac{1}{4}$, $F = \frac{2}{5}E$. (Poisson. Compare also Clebsch, *Theorie der Elasticität*, §§ 3 and 92.)

The moment of rotation exerted by the torsion of a wire may be calculated from F, if we imagine the wire to consist of thin concentric tubes. Let one of these tubes have the inner semi-diameter ρ, and the outer $\rho + d\rho$. On the surface of this tube let a straight line be drawn parallel to the axis of the tube. If we now twist the wire, so that the lowest section is turned through the angle ϕ, this line will be turned into a screw line, which has the inclination $\frac{\phi\rho}{l}$ to the vertical.

This is also the angle of displacement δ of the layers of which we have previously spoken. Therefore the torsional elasticity of the lowest section of the tube $2\pi\rho d\rho$ will seek to turn back to its original position with a total force $F\frac{\phi\rho}{l}\,2\pi\rho d\rho$. Since ρ is the radius of the tube, this force produces the moment of rotation $2\pi F\frac{\phi}{l}\rho^3 d\rho$.

Such a moment of rotation is, however, experienced by each tube in its end-section, so that the total moment of rotation of a wire of length l, and radius r, with an angle of torsion ϕ, equals

$$2\pi F\frac{\phi}{l}\int_0^r \rho^3 d\rho = F\frac{\pi r^4}{2l}\phi$$

With the help of App. 9 and 10 this gives directly the period of oscillation t, bearing in mind that if the forces are expressed by weights as in elasticity, the moment of rotation must be multiplied by the factor g.

37.—Determination of the Velocity of Sound by Dust Figures (Kundt).

The velocity of sound in dry atmospheric air at 0° is 331 m./sec., but in dry air at temperature t it is $331\sqrt{1+0{\cdot}00367t}$, and with ordinary humidity and moderate temperature approximately $u = 331\sqrt{1+0{\cdot}004t}$ m./sec. (see **15**).

This number may be used to determine the velocity of

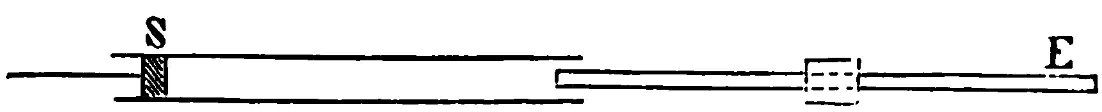


Fig. 18.

sound in a rod or tube rubbed longitudinally. The rod is laid horizontally and fixed in the middle. One end, E, is

rubbed longitudinally (p. 132), the other is inserted into a glass tube, at least 25 mm. wide, closed at the other end by a tight-fitting movable plug *S*. The tube must be well cleaned, and covered on the inside with lycopodium, silica, or sifted cork-dust. On rubbing the rod the impulses of the free end produce stationary air-waves in the glass tube, by which the powder is arranged in regular figures. By altering the position of *S* the place is easily found at which the agitation of the powder is most energetic, and at this the plug is left. The tube may also be permanently closed at *S*, and the whole slid back and forwards over the rod, instead of using a movable plug. A light cork or cardboard disc may be fixed to the end of a rod of small section in order to facilitate the communication of the impulses to the column of air. If afterwards the distance l between two nodes, that is half the wave-length, is measured by laying a divided scale underneath, and if L be the length of the rod which is rubbed, the velocity of sound in this latter is

$$u = 331\sqrt{1 + 0{\cdot}004t}\,.\,\frac{L}{l}\ [meter];$$

and therefore the ordinary modulus of elasticity (p. 128)—

$$E = \frac{u^2 s}{9810}\ \left[\frac{kg.\text{-}wt.}{mm.^2}\right]$$

where s denotes the density of the rod.

To obtain the length of the waves as accurately as possible, the distance of two nodes, distant several (n) wave-lengths from each other, is measured and divided by n. On the calculation by the measurement of a greater number of nodes, compare **3**, II.

Example.—A glass rod 900 mm. long gave, at the temperature 17°, a distance between two nodes of 62·9 mm. The velocity of sound in the glass was therefore

$$331\sqrt{1 + 0{\cdot}004\,.\,17}\,.\,\frac{900}{62{\cdot}9} = 4890 \text{ meters per sec.}$$

and the modulus of elasticity of the glass—

$$E = \frac{4890^2\,.\,2{\cdot}7}{9810} = 6580\ \left[\frac{kg.\text{-}wt.}{mm.^2}\right]$$

Comparison of the Velocity of Sound in different Gases.—The wave-lengths given by the same rod, rubbed as above described, in two different gases are obviously proportional to the velocities of sound in these gases.

We have the following general relations:—If

h is the pressure of the gas in mercury at 0°;
σ the specific gravity of the gas during the experiment;
σ_0 that at 0° at 0·76 m. pressure;
t the temperature;
c' and c the specific heats at constant pressure and constant volume (Table 36);
$g = 9{\cdot}810$, the acceleration by gravity;

then the velocity U is given by the following formula:—

$$U^2 = gh\frac{13{\cdot}596}{\sigma}\frac{c'}{c} = 9{\cdot}810 \times 13{\cdot}596 \times 0{\cdot}76\ \frac{1 + 0{\cdot}00367t}{\sigma_0}\frac{c'}{c}$$

$$= 101{\cdot}37\ \frac{1 + 0{\cdot}00367t}{\sigma_0}\frac{c'}{c}$$

These relations may serve either to calculate the velocity of sound in a gas for which σ_0 and c'/c are known, or *vice versâ* to determine from the observed velocity the density or the ratio of the specific heats.

37A.—Determination of the Number of Vibrations of a Musical Note.

(1.) *Graphically.*—The number of vibrations of a body giving out a note may be determined by allowing it to trace a curve on a moving smoked surface (*e.g.* a rotating drum) by means of a light flexible attached point. At the same time, some arrangement of known periodic time makes marks side by side with this curve. The number of waves between two or more time-marks are then counted. For the calculation compare **3**, II.

The marks may be made, for instance, by an electromagnetic marker, of which the circuit is closed at every swing of a seconds pendulum by means of a mercury cup; or the circuit is made through the primary of an induction coil, of which the ends of the secondary are connected, one with

the smoked drum and the other with a writing point near it. The sparks then mark the time.

A tuning-fork, of which the number of vibrations is already known, may be used to mark the drum along with the vibrations to be determined.

(2.) *By a Stroboscope.*—The velocity of revolution of a stroboscope disc is so regulated that the vibrating tuning-fork, string, spring, etc., when viewed through the disc with either naked eye, telescope, or microscope is apparently at rest. If several images at rest are seen, the velocity is reduced until one single image appears, or else the result is divided by the number of images. If the disc has m apertures and the number of revolutions $= k$ per second, the number of vibrations is $N = mk$. The number of revolutions is obtained either by the use of some counting apparatus, which is read during a measured time, or those of a more slowly moving wheel in the clockwork are counted, the ratio of the two numbers being known.

It is more exact and convenient to regulate the speed of revolution only so far as to leave a slow stroboscopic movement of the vibrating body. If then during the time t seconds s *stroboscopic* vibrations are observed, and in this time the disc has made S revolutions,

$$N = (mS \pm s)/t$$

The upper sign is to be chosen if, when the speed of rotation is increased, the stroboscopic vibration becomes slower, and *vice versâ.*

(3.) *With the Syren.*—A syren with clockwork counter is kept at the same pitch as the tone to be determined, and the rotations during a number of seconds are counted. By repeated observations tolerably trustworthy results are obtained.

(4.) *By Beats.*—Tuning-forks, or other sources of musical notes of periods nearly equal, or in a simple ratio, may be compared by the number of beats which they give together, each beat denoting an advance of a whole vibration by one of them. If it is doubtful which tone is the higher, one of them may be made a little flatter. If the beats become slower the tone so altered was the higher, and *vice versâ.* The pitch of

a tuning-fork may be lowered by gentle warming or by attaching a little wax.

(5.) *Monochord.*—A stretched string (monochord) of the length l m., of which 1 m. weighs p, stretched by a weight P, gives a primary tone of the number of vibrations

$$N = \frac{1}{2l}\sqrt{\frac{9{\cdot}81\,P}{p}}$$

The elasticity of the string itself makes the number of vibrations somewhat greater. Thin brass wire is the most suitable.

CAPILLARITY AND FRICTION.

37B.—DETERMINATION OF A COEFFICIENT OF CAPILLARITY.

The coefficient of capillarity may be defined as the weight of fluid which is supported by the unit of length of the line of contact of its surface with a thoroughly wetted plate. Or what is the same thing, if according to the law of Laplace the pressure of cohesion d arising from the curvature of a surface be given by the least and greatest radii of curvature r_1 and r_2 as

$$d = a\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

then a is the coefficient of capillarity.

In practice it is usual to express the lengths in mm., the forces or weights supported in mgr. The coefficient of capillarity is then given as [*mgr.-wt./mm.*] In passing over to the absolute system, the factor 10 is introduced from the reduction of mm. to cm., that of $\frac{1}{1000}$ from mgr. to grm., finally from gram-weight to dyne the factor $g = 981$. The coefficient of capillarity $[a]$ in this system is therefore 9·81 times larger than the a usually given.

I. *From the Rise in a Capillary Tube.*

A narrow tube of circular section is carefully cleaned and then immersed for a long time in the fluid to be investigated, so that it is thoroughly wetted by it. Water and many solutions in water are specially hard to produce actual wetting with. The capillary tube is then placed vertically so that a column of fluid which does not reach the top of the tube

remains raised in it. Let the height of this column be H. If s be the specific gravity of the fluid and r the inner radius of the tube in mm. the coefficient of capillarity is found as

$$\alpha = \tfrac{1}{2} rHs \ \textit{mgr.-wt./mm.}$$

H must be large in comparison with r. The height H must be taken as $\frac{1}{3}r$ above the lowest point of the meniscus.

Proof.—Since the interior circumference of the tube is $2\pi r$ and the amount of fluid raised is $\pi r^2 Hs$, the quantity of fluid which is supported by the unit of length of the circumference of the tube is $\frac{1}{2}rHs$, or since the radius of curvature of the hemispherical surface is r, the pressure (negative) due to the curvature is $d = \alpha . 2/r$, which must be equal to the negative hydrostatic pressure Hs. By another and older definition the product rH is called the coefficient of capillarity $= a^2$. These two coefficients are in the proportion $s : 2$.

Cf. Quincke, *Pogg. Ann.* clx. 341, 1877.

Radius of the Tube.—If a thread of mercury, which at temperature t occupies l mm. of the tube, weighs m mgr. we have in mm.

$$r = \sqrt{\frac{1}{\pi}\frac{m}{l}\frac{1 + 0{\cdot}00018t}{13{\cdot}6}} \quad \text{or for } t = 15^\circ \ r = 0{\cdot}1532\sqrt{\frac{m}{l}}$$

It is only the radius of the *upper end* of the capillary column which is wanted, so that the length of the mercury should be measured when the middle of it coincides with that point.

II. *From the Height of an Air Bubble or a Drop of Liquid* (Quincke).

(1.) *Air Bubbles.*—The fluid must be contained in a trough with a vertical plane glass side. In the fluid is produced under a horizontal plate immersed in it a broad air bubble, which should have a diameter of 20 mm. or more. If h be the vertical distance from the flat, lowest surface of the bubble to the plane of widest horizontal extension we have

$$\alpha = \tfrac{1}{2} s . h^2$$

For a correction which reduces the observation to an infinitely large bubble *cf.* Quincke, *loc. cit.* p. 354.

(2.) *Drops.*—A fluid, forming a drop upon a flat support which it does not wet, may be investigated by means of this drop in exactly the same manner. Here h denotes the vertical distance of the top of the drop to the point of greatest horizontal extension. The method can be used for melted metals of which drops have solidified on a warm plate. The heights are measured with a cathetometer (*cf.* Quincke, *loc. cit.*) or with a spherometer, in which the end of the screw is provided with a fine point or with a small horizontal disc.

Angle of Capillarity.—If in addition the whole depth of the bubble, or height of the drop h' is known, the marginal angle Θ between liquid and plate is obtained by

$$\cos\frac{\Theta}{2} = \frac{1}{\sqrt{2}} \cdot \frac{h'}{h}$$

III. *From the Length of Ripples* (L. Matthiesen).

Waves on the surface of a fluid are propagated partly by gravity, partly by the surface tension. If the wave-length be λ and the number of vibrations N, the velocity of propagation of the wave, u or $N\lambda$, is, according to W. Thomson, given by

$$u^2 = N^2\lambda^2 = g\left(\frac{\lambda}{2\pi} + \frac{a}{s}\frac{2\pi}{\lambda}\right)$$

In the case of very short waves, a few mm. in length, the first member may be neglected, and we obtain $N^2\lambda^2 = ga \,.\, 2\pi/\lambda s$, putting therefore $g = 9810$ mm./sec.^{2}, we have

$$a = \frac{1}{2\pi}\, s\, \frac{\lambda^3 N^2}{g} = \frac{1}{61600}\, s\lambda^3 N^2 \ mgr.\text{-}wt./mm.$$

Two light rods, which are fixed to the ends of a tuning-fork of known N (**37A**; N from 125 to 250), are brought into contact with the surface of the liquid and the fork is sounded. Stationary half-waves are then set up between the points, for which λ may be measured with compasses or scale.

Cf. Matthiesen, *Wied. Ann.* xxxviii. 118, 1889, where T is put for a/s.

IV. *By Dropping.*

The lower end of a wire of radius r mm. (p. 131) is melted in a small flame of as low a temperature as possible until the melted drop falls off. If the drop weighs m mgr.

$$a = \frac{m}{2r\pi}\ [mg.\text{-}wt./mm.]$$

The method is plainly subject to many sources of error.

Quincke, *Pogg. Ann.* cxxxiv. 365, 1868.

37c.—DETERMINATION OF THE COEFFICIENT OF FRICTION OF A LIQUID BY OUTFLOW THROUGH A CAPILLARY TUBE (Poiseuille, Hagenbach).

The coefficient of friction η is defined as the force which opposes the movement of a layer of liquid of unit surface in steady movement of velocity 1 at the distance 1 from a fixed layer, keeping parallel with it. The lengths are usually measured in cm., the forces technically in grm.-wt. Expressed in the absolute system the coefficient of friction is therefore $g = 981$ times larger than in the technical definition.

A volume of fluid v escapes in time τ, through a capillary tube of length l and radius r or section q (**19A**) under the constant pressure p

$$v = \frac{1}{\eta}\frac{\pi}{8}\frac{r^4}{l}p\,.\,\tau \quad \text{or} = \frac{1}{\eta}\frac{1}{8\pi}\frac{q^2}{l}\,.\,p\,.\,\tau\,.$$

η is the coefficient of friction, $1/\eta$ may be called the "fluidity" of the liquid.

(1.) *Determination of* η.—A capillary tube with circular, cylindrical bore is connected with a reservoir containing the liquid. The free surface of the latter is maintained at the constant height h. In order to avoid the opposing force of the surface tension of a drop the exit from the tube takes place most suitably in a wider vessel. h is then the difference of the heights of the free surfaces in the two vessels. If h is not constant the mean height during the experiment is taken. With the

specific gravity of the liquid $= s$ the pressure amounts to hs. If in τ sec. the volume v escapes we have

$$\eta = \frac{\pi}{8}\frac{r^4 hs\tau}{lv} \quad \text{or} = \frac{1}{8\pi}\frac{q^2 hs\tau}{lv} \; grm.\text{-}wt.\ sec./cm.^2$$

In the absolute cm.-grm.-sec. system, therefore

$$[\eta] = \frac{981}{8\pi}\frac{q^2 hs\tau}{lv}\;[cm.^{-1}.\, grm.\ sec.^{-1}]$$

(2.) *Relative Determination.* — Let a capillary tube be permanently connected with a reservoir, and allow the liquid at each experiment to escape during the fall of the surface between two marks. Let τ_1 and τ_2 be the times needed for this, s_1 and s_2 the specific gravities of the liquids, then the coefficients of friction are in the proportion

$$\eta_1 : \eta_2 = s_1\tau_1 : s_2\tau_2$$

For water at temperature t, lengths and volumes measured in cm., time in sec. (O. E. Meyer, Grotrian).

$$\eta = \frac{0{\cdot}00001091}{1 + 0{\cdot}0249(t-18) + 0{\cdot}000132(t-18)^2}$$

From this, by making a comparison with water, the friction can also be determined absolutely.

Correction for Velocity.—The above formulæ are for the case of sufficiently narrow or long tubes, *i.e.* for small velocity of flow. If the energy of movement of the liquid is an appreciable fraction of the work done by the force due to the pressure the value of η found as before must be multiplied by

$$1 - \frac{v^2}{2^{\frac{1}{3}}\pi^2 g r^4 h\tau^2} = 1 - \frac{0{\cdot}000082\,.\,v^2}{r^4 h\tau^2}$$

g is the acceleration of gravity. For cm. and sec. $g = 981$, hence the factor 0·000082.

Cf. Hagenbach, *Pogg. Ann.* cix. 385, 402, 1860; O. E. Meyer, *Wied. Ann.* ii. 387, 1877.

LIGHT.

38.—Measurement of an Angle of a Crystal by Wollaston's Reflecting Goniometer.

The instrument is so placed that its axis is parallel to a distant horizontal mark, *O*, such as a window-frame or roof-ridge, perpendicular to the line of sight. We assume that the edge of crystal has been already fixed, according to the instructions given on the next page, parallel to the axis. Holding, now, the eye close to the crystal, the observer turns the axis until the image of the upper mark, as seen in one of the crystal-faces, coincides with a lower horizontal mark, *U*, seen directly. The edge of the floor or the reflected image of the upper mark, as seen in a properly inclined mirror fastened behind the goniometer, may be used for this purpose. The position of the index (vernier) is then read off on the graduation of the circle. Then the circle with the crystal is turned until the image of *O*, as reflected in the other face of the crystal, coincides with *U*, and the index again observed. The angle through which the circle has been turned is the supplement of the angle made by the faces.

For accuracy in the measurement of the angle there is usually, inside the axis on which the divided circle turns, a second axis concentric with the first. On the use of this axis for the repetition of the measurement of the angle see **88.**

Adjustment of the Edge of the Crystal Parallel to the Axis of Rotation.

Two axes of rotation, perpendicular to each other, would be sufficient to give the edge to be measured any position

(Wollaston's original arrangement). But in this case the desired adjustment can only be attained by trial. If, however, a third axis of rotation be added, the edge to be

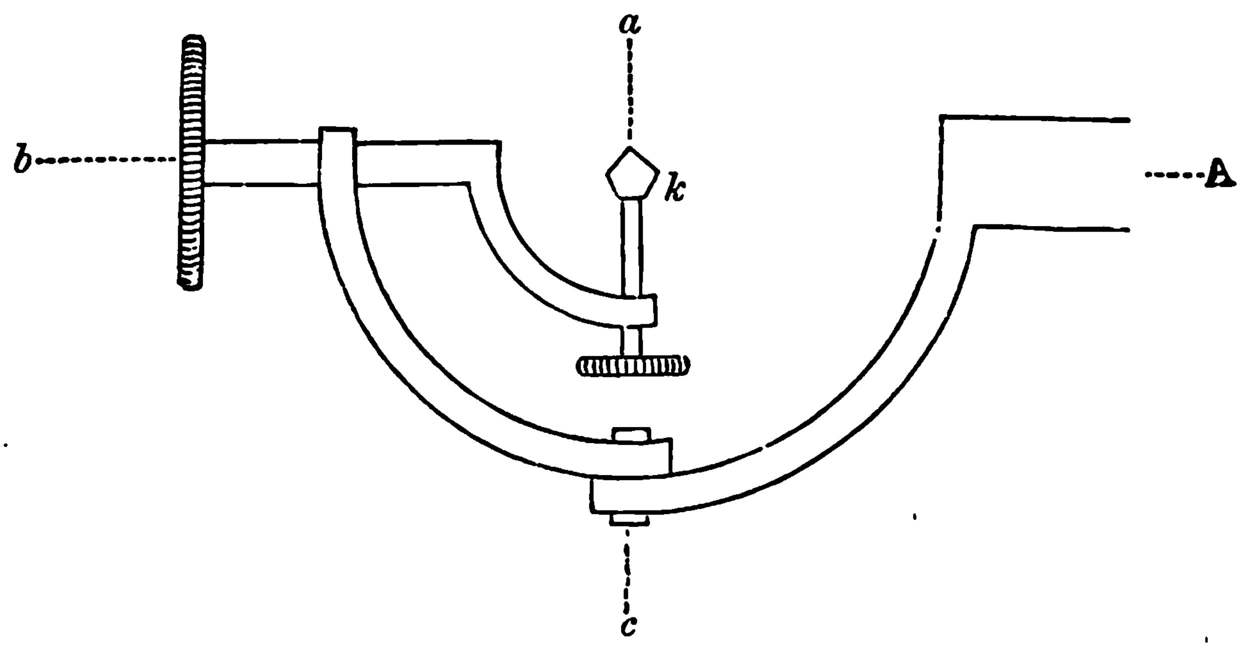


Fig. 19.

measured may be made parallel in a regular manner (*Naumann*).

A is the axis of the circle; *a b c* the axes for the adjustments; *k* the crystal held upon a piece of wax.

(1.) By turning round *c* a position is found in which the axes of A and *b* coincide, *i.e.* in which turning A does not disturb the milled head *b* from its place. Then by turning *a* the face I. of the crystal is placed parallel to A. (See below.)

(2.) The axis *c* is turned through an angle of from 60° to 90°; the face I. will usually be found to have altered its position with regard to the axis A. By turning *b* it is again placed parallel to A. The face I. is by this means made parallel to A and *b*, and therefore perpendicular to *c*; turning *c* will then no longer affect the position of I.

(3.) By turning *c* the face II. is made parallel to A.

In each successive adjusting of an axis, those already brought into position must not be altered.

In order to tell whether a face is parallel to the axis A, two points in the upper and lower marks are noted, one of which is perpendicularly under the other in the plane of the divided circle. If one of the horizontal window-bars has been used as a mark, it will be most convenient to make use of one of the vertical bars, and for the lower point that at which a plumb-line hanging from it cuts the lower mark.

With the roof-ridge a chimney is chosen, or a lightning-conductor, and underneath, its image in the fixed mirror. The face of the crystal is parallel to the axis, as soon as by a suitable rotation round A the image in the face of the upper point is made to coincide with the lower one.

On the measurement of angles by reflection compare also **39**, I.

39.—Determination of a Refractive Index with the Spectrometer (Goniometer).

General Rules.

(1.) *Collimator.*—The slit of the instrument must always, as seen through the lens belonging to it, represent an object at an infinite distance. To obtain this the telescope must be focussed for parallel rays. For this purpose the cross-threads of the telescope are first focussed clearly by moving the eyepiece or the cross-threads. The telescope is then pointed to a very distant object and drawn out so that the image of this object has no parallax with respect to the cross-threads,—that is, that they do not move over each other on moving the eye from side to side. If the cross-threads can be illuminated, their own image in a plane-mirror may be used instead of a distant object. Compare also No. 8 of this section. Finally the telescope is directed to the slit, and the tube carrying the collimator lens so drawn out that the image of the slit shows no parallax with the cross-threads. It then represents an object at an infinite distance.

(2.) *Illumination of the Cross-Threads.*—This illumination is effected by an inclined plane glass plate placed between the eyepiece and the cross. Rays of light fall upon the glass plate from a flame placed at one side, and are thence reflected to the cross and the object glass. If the telescope is adjusted for infinitely distant objects, the rays which come from a point of the cross emerge from the object glass as parallel rays, and give, if reflected again into the telescope—say from the face of a prism—a distinct image of the cross-threads. If this image coincides with the cross the line of vision is perpendicular to the reflecting plane.

(3.) *Reading the Circle.*—The provision of two opposite

verniers on a divided circle has not only the object of lessening the errors of reading, but specially of eliminating any accidental eccentricity of the graduation with regard to its axis. We must therefore at each observation read both verniers, which need not at all necessarily be distant exactly 180° from each other; but to avoid future uncertainty, it must be noted to which vernier each reading belongs. The required angle of rotation may be found by taking the mean of the angles given by the two verniers, or somewhat more conveniently by reckoning the degrees always by vernier I., and only taking the mean in the fractions (minutes) before subtracting the readings.

(4.) In order to prove whether the line of sight of the telescope is perpendicular to its axis of rotation, illuminated cross-wires in the eyepiece are employed. A small plane parallel-sided glass plate, silvered on both sides (**48**), is placed on the table of the instrument, best on a little stand with levelling screws, otherwise, fixed with wax, and so adjusted that the image of the cross-wires, as seen through the telescope, coincides with the wires themselves. It is obvious that on turning the telescope 180° the cross must again coincide with its image if the telescope is perpendicular to its axis of rotation. If this is not the case, half the deviation must be corrected by inclining the mirror, and half by inclining the telescope, and the proof again attempted, and so on till it succeeds.

If glass with quite parallel surfaces cannot be obtained, the plate must be cut so that the images are side by side, when it can still be used for testing the spectrometer.

(5.) That the axis of rotation of the table is perpendicular to the line of sight of the telescope is proved by turning the table with the mirror through 180° after adjusting the image of the cross-threads, when the images should again coincide.

(6.) If the mirror itself is provided with a little stand with levelling screws, it may, lastly, be used in an obvious manner to prove whether the plane of the table is parallel with the line of sight of the telescope.

(7.) To make a reflecting surface (prism-face, etc.) parallel with the axis of rotation of the instrument, the illuminated cross-threads of the adjusted telescope may be used as above

described, or the slit may be made to serve the same purpose as follows. First, the adjusted telescope is directed to the slit, and that part of the slit which coincides with the cross-threads of the telescope is marked by a horizontal thread. Then, when the image of the slit is observed in the reflecting surface, the cross-threads must appear at the same height if the surface is parallel to the axis of the instrument.

If two surfaces of the same body (prism) are to be adjusted, one of the faces is set at right angles to the line joining two of its three levelling-screws. This face is then first adjusted by these screws, and then the other by the remaining screw, the two first being left unaltered during the second operation.

(8.) *Testing the Parallelism of the Surfaces of a Glass Plate.*—This may be tested by the telescope with illuminated cross-wires as follows. (1.) By suitable focussing the reflected image of the cross-wires must appear clear, single, and undistorted. (2.) When by focussing the eyepiece all parallax between the cross-wires and their reflected image has been removed, it should be equally absent in the reflection from the opposite side of the glass plate. In this case the telescope is at the same time focussed for an infinite distance.

Determination of the Refractive Index.

The body of which the index of refraction is to be measured must have the form of a prism, which is got, in the case of a solid by grinding, in the case of a liquid by pouring it into a hollow prism with plane-parallel glass sides. The problem divides itself into two parts: the measurement of the angle of the prism, and the deflection of the ray of light.

I. *Measurement of the Angle ϕ of the Prism.*

(*a.*) *When the Telescope of the Spectrometer is fixed, and the Circle is movable.*—The prism is so placed that by rotating the circle one of the surfaces takes approximately the former place of the other. By means of the footscrews of the levelling-stand upon which it is placed, the prism is adjusted with its refracting edge parallel to the axis of rotation of the circle, as

described in No. 7. Then, by turning the circle, the image of some distant vertical object, or of the slit or illuminated cross-wires of the spectrometer, reflected from one face of the prism, is made to coincide with the cross-wires, and the position of the circle is read with the vernier. The same is repeated with the other face; the difference of the two readings, regard being had to any passing of the zero-point of the graduation, subtracted from 180° gives the required refracting angle of the prism ϕ.

(*b.*) *When the Prism is fixed and the Telescope movable with the Vernier or the Circle.*—The prism is placed with its refracting edge towards the slit, so that the line bisecting it would approximately pass through the slit or some distant object. The cross-wires of the telescope are made to coincide with the image of the object or slit reflected from the two prism-faces successively, and the difference of the readings is double the refracting angle.

The object must be at such a distance that the size of the prism is of no account in comparison with it. If the slit be used its tube must be so drawn out (section 1) that the rays from it passing through the lens fall parallel on the prism, so that it may appear as an infinitely distant object. The illuminated cross of the telescope also serves for the purposes of the measurement by making the cross coincide with its image as reflected successively in the two faces. The angle through which the telescope is turned is the supplement of ϕ. It is obvious that the angle of a crystal can be measured by either of these methods if the instrument possess an arrangement for fixing and adjusting the crystal between the slit and telescope.

II. *Measurement of the Angle of Deviation.*

The direct adjustment of the telescope upon the slit gives the direction of the unrefracted ray. There are four methods by which the deviation of a ray which has passed through the prism, and thence the refractive index, may be found.

(*a.*) *Minimum Deviation* (Fraunhofer).—The prism and telescope are so arranged that the refracted ray appears in the telescope, the prism is then slowly rotated, following the

movement of the image by the telescope. The prism is fixed in that position which produces the minimum deviation (that is, where the image moves in the same direction whether the prism be turned to the right or left), the cross-wires and the image of the slit are made to coincide, and the circle is read off. This reading, subtracted from that when the telescope is pointed directly to the slit, gives the angle of deviation δ. It is better to determine δ at the minimum deviation on both sides of the direct line of the collimator and take the half of the difference between the readings of the telescope in the two positions. The refractive index μ is then calculated, calling the refracting angle of the prism ϕ, by the formula

$$\mu = \frac{\sin \frac{1}{2}(\delta + \phi)}{\sin \frac{1}{2}\phi}$$

Proof.—The minimum deviation of a ray passing through a prism is produced when the ray makes, within the prism, equal angles with the two refracting faces, and therefore also with the two normals. These latter angles are $\frac{1}{2}\phi$ (see fig.). Let the angle of incidence, and therefore of emergence, from the prism be a, then by the law of refraction $\sin a = \mu \sin \frac{\phi}{2}$. The angle of deviation of the ray is $\delta = 2a - \phi$, therefore $\sin \frac{1}{2}(\delta + \phi) = \sin a = \mu \sin \frac{\phi}{2}$, from which the formula given above follows.

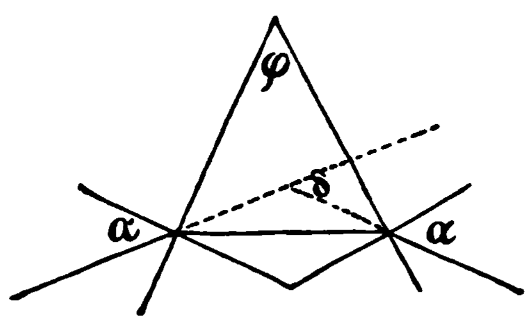


Fig. 20.

(*b.*) *Position of Perpendicular Emergence* (Meyerstein).—The prism is placed with the face which is turned towards the telescope perpendicular to it, *i.e.* so that the reflected image of the cross-wires coincides with the same as seen directly. The method assumes that the cross-wires can be illuminated. If we have, again, the angle of deviation δ (fig.), and the refracting angle of the prism ϕ,

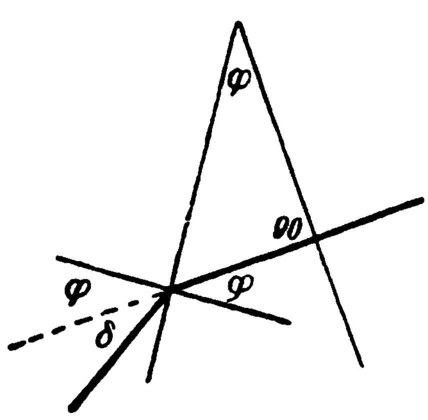


Fig. 21.

$$\mu = \frac{\sin (\delta + \phi)}{\sin \phi}$$

(*c.*) *The Ray reflected on its own Path* (Abbe).—The telescope, with illuminated cross-wires, is first arranged perpendicular to a face of the prism and the reading taken. The position is then found at which the rays from the cross-wires which have entered the prism by this face, and been reflected by the other again, emerge and fall on the cross-wires (the adjustment is by the image reflected in the further prism face). The two positions of the telescope make the angle ϵ with each other. Then

$$\mu = \frac{\sin \epsilon}{\sin \phi}$$

This follows from the figure to (*b.*), when $\phi + \delta = \epsilon$. *Cf.* Abbe, *Apparate zur Bestimmung des Brechungsvermögens*, Jena, 1874.

(*d.*) *Critical Incidence* (F. Kohlrausch).—A broad pencil of light falls "critically" on one of the faces of the prism (I. in the figure), *e.g.* from a flame coloured with sodium placed in the plane of the face. Looking through the other face of the prism the light is then seen sharply cut off. The telescope is adjusted to this boundary between light and dark. Let the angle which this direction S makes with the normal Z to the face II. amount to a.

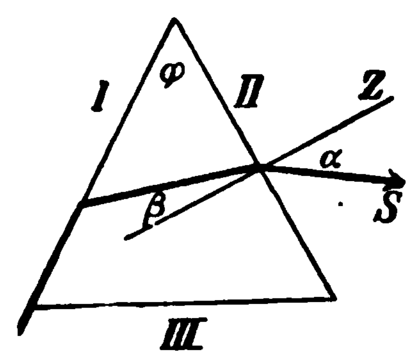


Fig. 22.

When the angle of the prism is diminished beyond an angle dependent upon μ the limiting ray S passes to the other side of Z, and a must then be reckoned negative.

With the illuminated cross-wires a can be measured direct by making the adjustment to the normal Z by (2).

In the absence of these the method is, after observing through face II., to observe through I. with the light entering II. For this purpose either the prism may be rotated leaving the telescope fixed, or the telescope rotated round the prism until it again reaches the limiting position between dark and light. Let this angle of rotation reckoned round face III. be called w, then

$$a = 90° - \tfrac{1}{2}(w - \phi)$$

and μ is calculated by the formula

$$\mu^2 - 1 = \left(\frac{\cos \phi + \sin a}{\sin \phi}\right)^2$$

For (see fig.) $\mu = sin\ a/sin\ \beta$, and from the fact of the critical incidence on I., $\mu = 1/sin\ (\phi - \beta)$. The elimination of β from the two equations gives the expression as above. *Cf.* F. Kohlrausch, *Wied. Ann.* xvi. p. 606, 1882.

The methods under (*c.*) and (*d.*) require no slit, but are mostly only available in homogeneous light. Method (*d.*) also requires no illuminating arrangement in the eyepiece of the telescope, and can be carried out with any telescope, and a graduated circle.

(*a.*) and (*d.*) are available for glass prisms with angles up to 70° or 80°, though one of about 60° is advisable. (*b.*) and (*c.*) can only be used for prisms of at most 40° in the angle, and in the case of the more highly refracting glasses only to 35.°

The index of refraction must of course refer to light of one particular colour. In sunlight, which can be thrown horizontally on the slit by means of a heliostat, Fraunhofer's lines are used. The figure contains the most important of these in the visible part of the spectrum, approximately in their positions in the prismatic spectrum. For the sake of easy remembrance, it may be noted that *ADFGH* lie at nearly equal intervals from each other.

In order to see *A* and *a* the slit must not be narrow, and

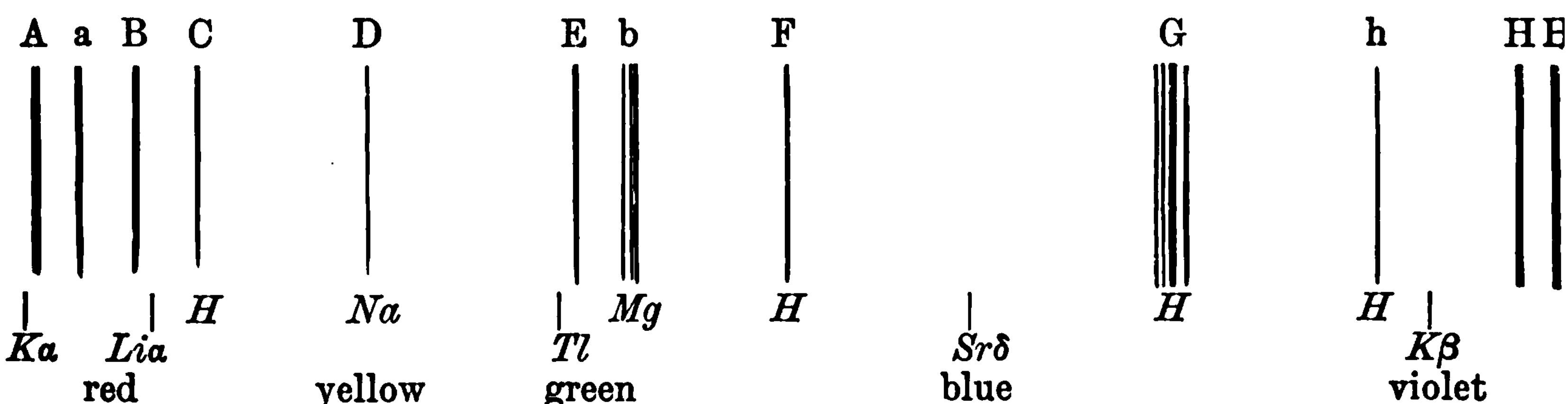


Fig. 23.

a piece of red glass should be held before it. With a narrow slit and greater magnifying power *D* is seen to be a very closely double line.

Failing sunlight the line *A* can be furnished by the potassium flame, *D* by the sodium flame, *F* a line in front of *G* and *h* by the electric spark, in a narrow Geissler's tube filled with rarefied hydrogen. The thallium and lithium lines also which do not coincide with any of Fraunhofer's are

together with *D*, and *Srδ* (which, however, is but feeble), well adapted for the definition of refractions. Cadmium and zinc (Table 19A) give in the electric arc a large number of lines in the ultraviolet spectrum. To make the ultraviolet light visible a "fluorescent eyepiece" is used, which contains, in the position of the cross-wires, a fluorescent screen of gelatine or uranium glass. Ultraviolet light is absorbed by glass to a considerable extent, so that the use of quartz for the prisms and objectives is advisable.

The difference of the indices of refraction for two particular colours (*e.g.* for *B* and *H* Fraunhofer's lines), is called the dispersive power for these colours.

To reduce an index of refraction measured in the air to its equivalent *in vacuo*, it must be multiplied by 1·00029, the index of refraction for light passing from a vacuum into atmospheric air.

See Tables 19, 19A and 20.

The index of refraction μ is obtained as a function of the wave-length λ by the series

$$\mu = A + B/\lambda^2 + C/\lambda^4 + \dots$$

39A.—Index of Refraction of a Plate under the Microscope.

Let the plate have the thickness d and the index of refraction to be determined μ.

Seen through the plate an object appears nearer than in its absence by $a = d(\mu - 1)/\mu$.

For if in the two, really *very* acute, right-angled triangles which have e as their small perpendicular side, we assume the hypothenuses as sensibly equal to the other sides d and $d - a$, we have for the sines of the angles of incidence and refraction respectively e/d and $e/(d - a)$. We have therefore

$$\mu = d/(d - a) \quad \text{or } a = d(\mu - 1)/\mu$$

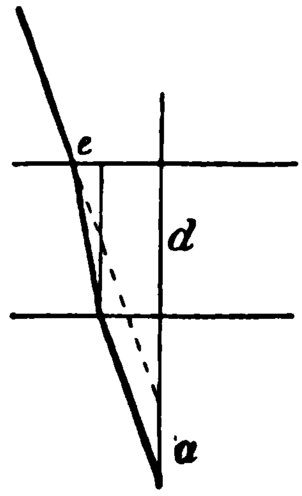


Fig. 24.

(1.) A microscope is focussed accurately upon an object. When the plate is introduced between the object and the objective the distance between them must be increased by a certain amount a in order that the sharp focus

may be restored. The index of refraction of the plate is then

$$\mu = \frac{d}{d - a}$$

(2.) Distinct marks are present on both the upper and under sides of the plate. In order to change the focus from the one to the other let a movement h be necessary. Then, as is easily deduced from the previous formula,

$$\mu = \frac{d}{h}$$

(3.) A plainly visible mark is placed on the upper surface of the plate with white paint, and the microscope focussed upon it. Now, in order to see the image of the mark reflected from the under surface of the plate the distance between objective and plate must be diminished by a. The index of refraction of the plate is

$$\mu = \frac{2d}{a}$$

In method 3 the observation is made in a top light, and the under side of the plate is blackened, or better silvered (**48**).

To determine the amount of the displacements with accuracy the fine adjustment of the microscope may be used, if the value of the screw thread is known and the head of the screw is graduated.

The adjustment is most accurately made with cross-wires in the eyepiece, which show no parallax with respect to the image. An objective of short focus and not too great aperture is most suitable. With thick good plates even the third decimal may be relied upon.

On the determination of the index of refraction of a liquid from the axial angle of a crystal, see **47**, end.

40.—Determination of the Refractive Index from the Angle of Total Reflection (Wollaston).

When a ray of light moves in a medium of index of refraction μ, and reaches the limiting surface between this and a

second of smaller index of refraction μ_1, total reflection occurs as soon as the angle of incidence on the surface becomes greater than $\sin^{-1}\frac{\mu_1}{\mu}$. Hence the observation of the limiting angle Φ of total reflection gives the equation

$$\frac{\mu_1}{\mu} = \sin \Phi$$

from which, if the index of refraction of one of the media is known, that of the other can be calculated.

This method of determining an index of refraction generally requires more simple instrumental means than those of **39**, and possesses the advantage that it can be used in the case of bodies not perfectly transparent. Any exact determination must of course refer to light of some particular colour (p. 156).

I. *With the Prism.*

(1.) *Index of Refraction of the Prism.*—One face I. of a prism is illuminated from the interior, *i.e.* through the third face with diffused (sodium) light, while the telescope of the spectrometer is directed towards face II. The limit of total reflection from I. then appears as a sharp boundary between bright and less bright. The telescope is adjusted to this boundary line. The direction of the telescope is then obviously the same as S in the figure to **39**, II. (*d.*), and the index of refraction of the prism is obtained exactly as there described, and by the same formula.

(2.) *Index of Refraction of another Body.*—The body is cemented with some highly-refractive liquid on to face I. of the prism, and the process of the preceding paragraph gone through. Deceptive light from other faces is shut off by blackening them.

If μ_1 be the index of refraction to be determined in the body, μ the greater one of the prism, ϕ the angle of the prism, a the angle which the direction of vision of the limit of total reflection makes with the normal to the face of the prism towards which the telescope is pointed (fig., **39**, II. *d.*) we obtain

$$\mu_1 = \sin \phi \sqrt{\mu^2 - \sin^2 a} - \cos \phi \, . \sin a$$

For $\mu = \mu_1/sin\,(\phi - \beta) = sin\,a/sin\,\beta$ from which the expression follows.

Cf. F. Kohlrausch, *Wied. Ann.* xvi. 607, 1882.

II. *With the Total-reflectometer* (*F. Kohlrausch*).

The body is so attached to the instrument that the axis of rotation lies in the plane of the reflecting surface (see below). The back of the body and its surroundings are blackened with Indian ink. A small glass vessel filled with some highly refractive fluid such as carbon disulphide, or bromnaphthalin (Groth), is then inverted over the body, and surrounded with very translucent tissue-paper (moistened, if necessary, with petroleum), which is lighted on one side with the sodium flame.

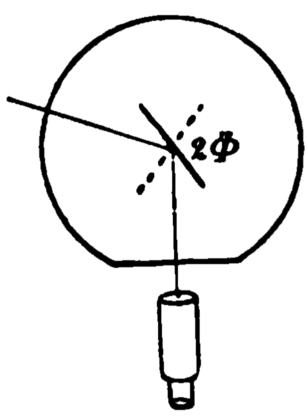


Fig. 25.

By trial, such a position of the lamp, and such an inclination of the reflecting surface, is found that the eye, through the telescope which has been focussed on an object at a great distance, sees the field of view in the face divided into a brighter and a dimmer half. By rotating the body the dividing line is brought into the line of sight, and the position of the verniers on the divided circle and the temperature are read off. The same process is then repeated on the other side of the glass vessel. The angle between the two positions is 2Φ, double the limiting angle of total reflection between the fluid and the body, therefore $\mu_1 = \mu\ sin\ \Phi$ if μ be the index of refraction of the liquid.

Adjustment of the Instrument.—The circle is rotated 90° from its principal position, so as to get a free sight, and the telescope is focussed for parallel rays (**39**, 1). To prove whether it is parallel to the plane of the divided circle, the telescope is directed to a well-marked distant point. The free line of sight to the same point must then lie in the plane of the circle.

The circle is now again brought into position and the body fixed to the carrier with cork, etc., so that the reflecting surface is brought to coincide with the axis of rotation and made parallel with it. This is known to be the case by the aid of a fixed mirror parallel to the axis. The image of the eye or of

a little flame must appear at the same height in the mirror and the face of the body.

The smaller or more imperfect the reflecting surface (as is generally the case in natural crystals), the greater care must be taken that the surface and the line of sight of the telescope actually pass through the axis of the instrument.

The refractive index of pure carbon disulphide is 1·6276 for 20° C., and diminishes for each degree of increased temperature about 0·00080. The temperature must therefore be carefully observed. A screen with a thick glass plate, placed before the flame, diminishes the heat derived from it. At the same time the screen serves to render the background of the little flask dark.

Crystals.—Doubly refracting objects in general give two limits corresponding to the two indices of refraction. Uniaxial crystals are most conveniently examined on a surface perpendicular to the principal axis (see **46**A). The horizontally polarised ray (that is, the ray disappearing at the vertical position of the greater diagonal of Nicol's prism) is the ordinary, the other the extraordinary, one. If the crystal surface is parallel to the optic axis, both indices are obtained when the optic axis is parallel to the axis of rotation. In this case the extraordinary ray is horizontally polarised.

A crystal surface, however placed, yields in all directions the ordinary ray; but each surface contains also a direction perpendicular to the optic axis (for instance, in the cleavage surface of a rhombohedron that of the bisecting line of the lateral angle; in the surfaces of the quartz pyramid, that of the bases of the triangles). If this be placed horizontally, the observation gives both indices of refraction.

If we have an optically biaxial crystal cut parallel to a principal plane (**46**A), we obtain two indices when an axis of optical elasticity is placed horizontally. A direction of the surface perpendicular to this yields the third refractive index, and one or other of the former ones.

Fluids.—(1.) In order to determine the index of refraction μ of the liquid in the little flask a little plane plate of known index of refraction (*e.g.* rock crystal with the indices 1·5442 and 1·5533 for sodium light) may be

used. Or an observation is made with a film of air behind a plate. Then

$$\mu = \frac{\mu_1}{sin\,\phi} \qquad \text{or with air } \mu = \frac{1}{sin\,\phi}$$

(2.) A drop of liquid behind a plane plate can be investigated just like a solid.

Cf. F. Kohlrausch, *Wied. Ann.* iv. 1. On the use of the total-reflectometer in white light with the help of a spectroscope see Pulfrich, *ibid.* xxx. 487, 1887. For another arrangement of the instrument with a cylinder of heavy glass instead of the highly refracting liquid, see also Pulfrich, *ibid.* 193.

III. *With the Refractometer* (Abbe).

Abbe's refractometer has a double prism P of heavy glass between the surfaces of which a drop of the liquid under examination is introduced. The prism has a motion of rotation round a fixed axis. To use the instrument it is turned over, one of the prisms pushed aside (cautiously! the soft glass is easily damaged) and after placing the liquid on the under surface, returned to its place. Rays of light reflected from the mirror S on to the prism are totally reflected if they fall on the layer of fluid with more than a certain degree of obliquity so that when the position of the prism is correct, the field of view of the telescope adjusted for parallel rays, is sharply divided if *e.g.* sodium light is used. The boundary is adjusted to the cross-wires which of course have been previously made clearly visible by focussing with the eye-piece. The division on the arc K then reads directly the index of refraction for sodium light.

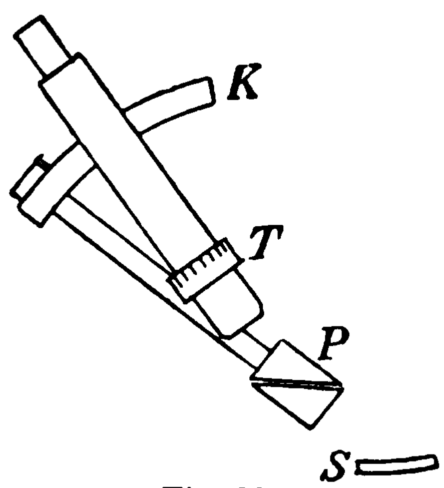


Fig. 26.

With ordinary white light the dispersion of the liquid is at once obtained as follows:—The field of view is here usually coloured. The compensator, that is the divided drum T, which rotates two direct vision prisms set in reverse positions, is then so placed that the coloured field gives place to a sharp boundary. The boundary is brought on to the cross-wires

and the readings on arc and drum are taken. A second position of the drum which gives a sharp division is found, the cross-wires adjusted, and readings again taken.

The mean of the two readings on the divided arc gives the index of refraction for sodium light, the dispersion is calculated according to a table supplied with each instrument.

A solid is fixed to the under surface of the upper of the two prisms by a drop of some liquid of high refractive index (oil of cassia, bromide of arsenic). Transparent bodies are illuminated by daylight or lamplight passing through them, other substances by light falling on them from the side. A few trials will make the boundary plainly visible.

As a test of the accuracy, or if necessary for the correction, of the graduation, fluids with known indices (Table 20) may be used, particularly water or a plate of a known glass or of rock crystal (*cf.* II.) The vernier must be very firmly connected to the prisms. For many fluids the temperature introduces uncertainty.

Cf. Abbe, *Apparate zur Bestimmung des Brechungsvermögens*, Jena, 1874; and *Sitz. Ber. d. Jenaischen Ges. f. Med. u. Nat.*, 1879, Feb. 21.

IV. *With the Spectrometer.*

The method described under II. may also be carried out with the spectrometer (**39**) if the plate under examination, while movable with the divided circle, can be enclosed by a fixed trough with a front of plane glass.

If the object consists of a transparent body in the form of a very thin large plane-parallel plate, the light from the slit may be employed. The trough must have two opposite plane walls.

Parallel light from the slit (**39**, 1) is allowed to fall perpendicularly upon one wall, and the slit observed through the body with the telescope. The two oblique positions of the body at which the slit (illuminated with homogeneous light) suddenly disappears, are 2ϕ apart. If a prism, most conveniently of direct vision, be brought between the trough and the telescope, and the slit be illuminated with sunlight, a spectrum (p. 156) appears. By turning the object-plate, the

limit of total reflection may be made to coincide with any line.

A box consisting of two plane-parallel plates fixed parallel to each other, with a film of air between them, if brought into the trough and treated in a manner exactly similar to that just described, gives at once, from the half angle of rotation ϕ, the index of refraction μ of the fluid in relation to air, as—

$$\mu = \frac{1}{\sin \phi}$$

Cf. E. Wiedemann, *Pogg. Ann.* clviii. 375, and Terquem and Trannin, *ibid.* clvii. 302.

40A.—Determination of Small Variations in the Index of Refraction by Interference (Jamin).

The "interference refractor" consists of two similar, thick (*e.g.* about 3 cm.) plane glass plates with parallel sides, which are most advantageously silvered on the back, placed facing each other and parallel, making an angle with the line joining their centres of from 45° to 50°. A ray of light falling on the first plate affords by reflection at the two surfaces two rays. After falling on the second plate each ray again divides into two, of which those figured emerge at the same point and in the same direction. These rays, if the plates are slightly turned from their parallel position, give rise to a system of interference bands. We may place in the path of the two rays two bodies optically similar. If now variations take place in the path of one of the rays, on account of changes of temperature, pressure, concentration, etc., while the system of bands is being watched, a movement is seen to take place among them. The alteration of the index of refraction corresponding to the alteration of the conditions may then be determined from this displacement, the length of the body producing it, and the wave-length of the light used (see under II.)

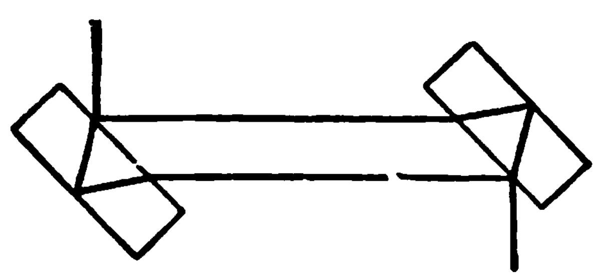


Fig. 27.

I. *Adjustment of the Apparatus.*

The two plates may stand upon a horizontal plane surface. They are either mounted together on a heavy stand, or else, to allow a choice in the dimensions of the bodies introduced between them, on separate stands, in any case, however, on a very firm support.

If a sodium flame, which it is best to surround with a black cylinder provided with a suitable opening, be placed about 150 cm. behind the first plate, the interference bands are seen either at once or after a slight rotation of the second plate on looking at this plate with the eye adjusted for distant vision. The rotation of the plate is continued until the bands appear sharply defined, parallel to each other and straight; their *direction* is in most cases of no consequence. It will, however, mostly be preferred to place them either horizontally or vertically (see below, and p. 167), which can always be easily effected by rotation round the two axes.

The best means of observation is a firmly fixed telescope, of which the object-glass is covered with a cap provided with a narrow opening placed perpendicular to the bands. If fractions of the breadth of the bands are to be measured by the telescope (a compensator, see the end of the section, may also be used for this purpose), a micrometer in the eyepiece is advisable.

After the introduction of the experimental tubes, etc., between the plates, and, if necessary, placing suitable screens to cut off the two outer images, the direction and distance of the bands and the telescope receive their final adjustment.

Too many bands in the field of view may easily give rise to errors in counting their movements, and some may, if needful, be screened off.

If it is necessary to carry out the observations in white light (see III.), wide *horizontal* (p. 167) bands are first of all produced, using sodium light. On substituting white light it is possible by very slow rotation of the second plate round its vertical axis to bring coloured bands into the field of view. The *colourless* band in the midst of this system is brought into

the centre of the field, and the telescope focussed for the greatest distinctness upon it.

Under some circumstances an arrangement of the instrument with parallel white light (sunlight) will be preferred (see Quincke, *Pogg. Ann.* vol. cxxxii. p. 50, 1867). The heating of the apparatus and the variations of the temperature of the place of observation produced when an intense source of light is used, may easily bring about a disturbing movement of the bands.

A trough with a solution of alum in front of the source of light, and confining the use of the light to the actual time of the observation, are in addition advisable even in the case of feeble lights.

II. *Measurement in the Case of Steady, Uniform Variations.*

If the changes in the conditions of a body, which produce the change of index of refraction in the substance, take place so continuously and uniformly in all parts of the body that the movement of the bands can be followed, as for example in the alteration of the pressure of a liquid, the gradual rarefaction of a gas by slowly pumping it out, etc.; the measurement is very simple. The number of bands is counted by which the system (in sodium light) is displaced relatively to the cross-wires of the observing telescope during the alteration of condition in the body, estimating the fractions of the width of a band or measuring them with the eyepiece micrometer.

If then

L = the length of the layer traversed,
λ = the wave-length of the light used (Table 19A),
s = the number of bands displaced,
μ_1 and μ_0 = the indices of refraction after and before the alteration,

we have

$$\mu_1 - \mu_0 = s\frac{\lambda}{L}$$

For if $\lambda_0 = \lambda/\mu_0$ and $\lambda_1 = \lambda/\mu_1$ the wave-lengths in the original and the altered state we have obviously

$$s = L/\lambda_1 - L/\lambda_0 = (\mu_1 - \mu_0)L/\lambda$$

III. *Measurement in the Case of Alterations not Uniform.*

The movement of the bands cannot always be followed (*e.g.* when a salt is dissolved in a liquid, or when a solution is replaced by a more concentrated one). The displacement of the bands cannot then be counted directly, but may be obtained as follows:—

Let white light be used. The coloured bands then produced afford a means, by the difference of their appearance, of numbering the corresponding bands in sodium light, which are not there distinguishable, by taking as zero a black band near the position of the colourless band seen with white light. The displacement of this zero produced by the change of conditions in the substance is found by counting the bands (as seen in monochromatic light) displaced by rotating the second plate on its vertical axis until the zero as seen with white light again appears on the cross-wires. We thus obtain, subject to a correction, the number of bands by which the monochromatic band system has been displaced. From this number the alteration of the index of refraction is determined as in II.

The bands must be horizontal in order to show in white light. The difference of phase which two interfering rays receive on passing through the refractor (see Verdet-Exner, *loc. cit.*) is

$$\phi = 2\mu d\ (\cos b - \cos b')$$

where μ is the refractive index, and d the thickness of the plates, and b and b' the angles of emergence from the first plate and entrance into the second respectively. If ϕ is to be equal to 0 we must have $b = b'$. This happens for any ray which travels in a plane with which the normals to the plates make equal angles, and which is at right angles to the plane in which they lie. The plates must therefore be inclined to each other in the vertical direction, but the bands are then horizontal.

The correction mentioned consists in the alteration of the number of bands observed as above by a whole number of band breadths, and owes its origin to dispersion.

Our measurement is effected by the compensation of two opposing differences of phase as we saw. This compensation is not possible for all the colours of white light at the same time, since the two opposing parts of the apparatus (refractor

and interposed body) do not introduce the difference of phase for the different colours in the same proportion. We obtain rather for that difference of phase resulting from the opposing actions of the two parts which has the same magnitude for all colours a colourless band.

If, for example, one of the parts introduces differences of phase in the red and blue in the proportion of 7 : 9·8 to each other, and the other in the proportion 7 : 10; on the introduction of 7·5 and 10·5 wave-lengths respectively in the first case, and − 7 and − 10 in the second, the resulting difference of phase in the case of red and blue will be equal and amount to + 0·5. For the other colours of the source of light, there will then usually ensue a nearly equal difference of phase, so that we obtain a black band, corresponding to the difference of phase 0·5.

This difference of phase which corresponds to the achromatic condition varies with the increase of the compensated difference of phase. The achromatic line therefore alters its position uniformly over the compensated system of bands obtained with sodium light moving gradually from one band to another, passing in the process through different stages of brightness from black to white, which correspond to the brightness of its position in the monochromatic system. When it falls on a dark band the coloured system of bands is arranged symmetrically with regard to a black line; when it falls on the middle of a light band the line is white. The zero point therefore of our counting of the bands moves slowly along the system.

Cf. Sirks, *Pogg. Ann.* cxl. p. 621; cxli. p. 393, 1870. The observation of this movement also gives the alteration of the dispersion (see Hallwachs, *Wied. Ann.* 1892).

To obtain the amount of this correction an auxiliary experiment is performed, in which the alteration in the body investigated is brought about in sufficiently small stages until the achromatic line appears black. The change is then continued further until the colourless line after passing through the white stage again appears black. From this is obtained the magnitude of the alteration in the conditions of the body, or the corresponding shift of the bands for which the colourless

line is displaced by the breadth of one band in the direction to be observed.

We can then, when in an experiment the total displacement of the bands is found, always say what was the original number of one of the dark bands near the achromatic line, *i.e.* how far distant this is from the zero first chosen.

If the change of condition consists in a change of concentration in a solution, the corresponding movement of the colourless line is fairly independent of the original concentration.

According as the zero is displaced by the alteration in the body in the direction of the movement of the bands, or in the contrary direction, the altered number of the dark band accompanying the achromatic line which is chosen for the adjustment must be subtracted from or added to it.

Since the fractions of the breadths of the bands are measured directly with the eyepiece micrometer, the observations with the system of bands in white light are only used to obtain the whole numbers of the displacement, our correction is therefore confined to a whole number of wavelengths.

Cf. Siertsama, *Der Jamin'sche Interferentialerefractor, Proefschrift,* Groningen, 1890; Hallwachs, *loc. cit.*

Jamin's Compensator.—This may be used with advantage in many cases to compensate the differences of phase instead of the rotation of the second plate (see Quincke, *loc. cit.* p. 204). If the compensator is to be used for white light, what has been said above as to the movement of the colourless line must be considered in its empirical graduation.

Cf. further on the general theory of the apparatus: Verdet-Exner, *Wellentheorie d. lichtes,* i. 94, 1891; Ketteler, *Farbenzerstrenung d. Gase,* p. 29, 1865; Zehnder, *Wied. Ann.* xxxiv. 91, 1888.

41.—Spectrum Analysis (Bunsen and Kirchhoff).

The apparatus for spectrum analysis requires, besides the telescope F and slit s previously mentioned as forming the

spectrometer (**39**), a third tube *R* with a micrometer scale. This is reflected in the face of the prism which is next the telescope.

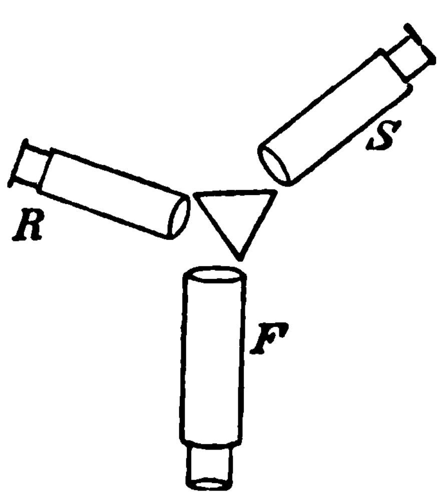


Fig. 28.

I. *Adjustment of the Apparatus.*

This is accomplished in the following manner, the *order* of the operations being specially observed:—

(1.) The slit must appear as a very distant object. If the right adjustment be indicated on the tube, the telescope need only be focussed so as to give a sharp image of the slit, otherwise the telescope must be focussed on some distant object, pointed to the slit, and the latter drawn out till it appears clear and sharp.

(2.) The prism must be adjusted to the position of minimum deviation. To attain this end, where the prism has not been fixed in the proper position by the maker, the slit is illuminated with the sodium flame, and the prism placed approximately in its right position before the collimating lens; and when the direction of the refracted ray has been found with the naked eye, the image of the slit is sought with the telescope. The prism is then turned (following the image, if necessary, with the telescope) until the image stops and begins to move backwards, and is then fixed in this position.

(3.) The reflected image of the scale should be clearly visible. It is illuminated by a lamp placed about 20 cm. from it. When, by turning the tube, the image is found, the tube is drawn out till the scale appears distinct. The images of the slit and scale should not alter their relative positions in the telescope on moving the eye before the eyepiece.

(4.) The sodium line should be made to fall upon some particular division of the scale—that adopted by Bunsen and Kirchhoff being the 50th. This adjustment is made by turning the tube carrying the scale, which should then be clamped.

II. *Valuation of the Scale.*

In order to know the points of the scale which correspond to the lines of the different chemical elements, their

flames should be observed separately, and the position on the scale (with their brightness, width, colour, and sharpness) of the lines should be noted. It is more convenient to employ for this purpose the copies which are published of Bunsen and Kirchhoff's maps, or Table 19, which is referred to the same scale. For this purpose the scale of the apparatus may be reduced to that of the charts in the following manner:—

The positions of a few known lines near the ends and in the middle of the spectrum (say *a*, *D*, *F*, *G*, *H*, in sunlight, or *Ka*, *Li a*, *Na*, *Sr* δ, *K*β, see p. 156), are observed on the scale of the instrument. The observed positions are laid down on square-ruled paper as abscissæ, and the corresponding positions on Bunsen's scale as ordinates, and a curve drawn through the points obtained. This will seldom differ much from a straight line. On this the position on Bunsen's scale, corresponding to any observed position on that of the instrument, will be found as the ordinate. In many spectroscopes the scale is made nearly to agree with Bunsen's. When this is the case, *Na* is made to coincide with the 50th division, and the scales are compared by a series of observations. The curve is more conveniently constructed, for the corrections only of the scale, by taking the differences from Bunsen's scale as ordinates in the graphical construction (see Table 19).

III. *The Analysis.*

The lines due to the bodies when placed in a Bunsen's gas-flame are observed on the scale, and the bodies identified by the agreement of the lines with those due to known substances. In doing this the following remarks must be attended to:— First, not only must the positions of the observed lines be noted, but, approximately at least, their brightness, width, and sharpness. For instance, *Sr* β and *Li a* fall very near together; but while *Sr* β is hazy, *Li a* is quite sharp. Bands may be represented graphically by making the intensity of the light at any point the ordinate at the point, and so draw a curve to represent the spectrum (Bunsen).

For distinguishing the alkaline earths, it is best to make use of the (faint) characteristic lines in the blue part of the spectrum.

The bodies are always placed in the front border of the flame on platinum wire, the glowing solid part so far down that it does not give any disturbing continuous spectrum. It is advisable to use, first of all, a narrow slit, in order to separate lines lying close together, and then to repeat the observation with a wider one, to detect lines of feeble brightness. Similarly it is well to employ, first, a small gas-flame for easily vaporisable bodies (*K*, *Li*), and then a larger one for those which are less so (*Sr*, *Ba*, *Ca*). The spectra of the latter often require a longer time before they make their appearance. The bodies are usually employed as chlorides; sodium and potassium, however, on account of the decrepitation of the chlorides, are more conveniently used as carbonates. The enfeeblement of the spectrum, in the course of a long experiment, is often due to the fact that the chlorides by ignition are converted into the less volatile oxides. The intensity of the light is in this case momentarily restored by moistening the bead with hydrochloric acid. Compounds, such as the sulphates of the alkaline earths, which are by themselves scarcely volatile, and are not decomposed by hydrochloric acid, are ignited in the lower reducing part of the flame and then moistened with hydrochloric acid.

The most effectual way of cleansing a platinum wire from substances which volatilise with difficulty, is dipping it into hydrochloric acid and pure water and long ignition in the point of the flame, or before the blowpipe.

Extraneous light must be carefully cut off,—by a black screen behind the lamp, by a cover for the prism, which only leaves a passage for the light through the three tubes, and, lastly, by a black paper shade hung from the telescope. The last renders unnecessary the wearisome closing of the eye which is not in use. The scale should never be more strongly illuminated than is necessary for recognising the divisions and numbers. In order to see very faint lines, the light passing through the scale may be entirely cut off.

The Bunsen's gas-flame itself gives a number of feeble lines, principally in the green and blue. In order that these may not mislead the observer, they should be previously observed and the strongest noted. The lower part of the flame,

too, in which these lines are the strongest, is not used for the observation. The sodium line also is seen in most preparations; indeed, the air itself frequently contains so much sodium that the reaction occurs without any visible cause for it being present.

Ultraviolet Spectrum.—This is investigated with the aid of a fluorescent eyepiece (p. 157), or else by projection on a fluorescent screen, or by photography. That no light may be absorbed, apparatus of quartz should be used, and reflection gratings instead of transmitting ones (Rowland).

Absorption Spectra.—The spectrum analysis of white light which has passed through coloured bodies, especially solutions, may also be of importance. Sharp lines are here but seldom produced.

On the measurement of the intensity of light in the spectrum, see **47A**, III.

42.—Wave-Length of a Ray of Light.

(1.) *Diffraction Gratings.*—Upon the table of the spectrometer (**39**) is placed a plate of glass ruled with a very fine grating of lines (Nobert's test-lines) parallel to the slit, the plate perpendicular to the tube carrying the slit, the engraved face turned towards the telescope. The telescope is first focussed on a very distant object, and the slit adjusted to this focus (**39**, 1). Using, then, homogeneous light, we shall observe, in suitable positions of the telescope, besides the middle bright image of the slit, a first, second, etc., fainter image on each side of the middle. Let l be the distance between the lines in the grating on the glass plate, δ_1, δ_2, δ_3 . . . the angles of deviation of the images from the middle one, the wave-length of the light used is—

$$\lambda = l \sin \delta_1 = \tfrac{1}{2} l \sin \delta_2 = \tfrac{1}{3} l \sin \delta_3, \text{ etc.}$$

For in each of these directions the distances from the separate spaces of the grating to the telescope differ from each other by whole multiples of a wave-length. The light-vibrations which reach the telescope (adjusted for parallel rays) in one of these directions are therefore in the same phase, and so

reinforce each and form an image. Any other direction contains diffracted rays at irregular distances from the separate spaces; the rays are therefore in very different phases, and so, when collected by the telescope, destroy one another.

The grating is known to be placed accurately perpendicular when corresponding images on each side have the least distance from each other.

Light not homogeneous is dispersed by the grating into spectra, in which, according to the formulæ given above, the light consisting of the longer waves (red) appears most deflected. In using sunlight in which the Fraunhofer's lines (p. 156) are used for the definition and adjustment of the colour, the first spectrum and the greater part of the second are pure; beyond this the spectra interfere with each other. In order to recognise the lines from the map on p. 156, it must be remembered that the interference spectrum appears more and more contracted the more the violet end is approached.

Reflection Gratings act in the same way as the above when the reflection of the light in the plane of the grating is taken as the source of light. *Cf.* Quincke, *Pogg. Ann.* cxlvi. 43, 1872.

(2.) *Newton's Rings.*—Let a spherical surface of large radius of curvature r (**43**) lie upon a plane surface. Let the ring, p in number, which is observed when viewed vertically, have the diameter a_1, the ring $(p+k)$ the diameter a_2. Then the wave-length of the light used is

$$\lambda = (a_2^2 - a_1^2)/kr$$

43.—Measurement of a Radius of Curvature.

I. *With the Spherometer.*

The spherometer (**18**, 7) is first adjusted upon a surface known to be a plane (**39**, 8), and then upon the surface to be measured. Let the positions of the central point in the two cases differ by a height h. Let, further, l be the side of the equilateral triangle formed by the three fixed points. The

measurement of l is most simply performed by pressing the points on to paper. The radius of curvature is

$$r = \frac{l_2}{6h} + \frac{h}{2}$$

For if H be the altitude of the triangle, with the side l we have

$$2rh = \tfrac{4}{9} H^2 + h^2. \quad \text{Since, further, } H^2 = \tfrac{3}{4} l^2$$

the above expression follows.

II. *By Reflection* (R. Kohlrausch).

The determination of the radius of curvature with the spherometer is limited to large surfaces. In order to determine that of a small surface, if reflecting, and not too little curved, we may proceed as follows:—

The object is fixed so that the surface to be measured is perpendicular, and two lights are placed at some distance in front of it, at the same height as the object, and at the same distance from it. Half-way between the lights is placed a telescope focussed upon the surface. Lastly, immediately in front of the surface, and parallel to the line joining the lights, is placed a small scale divided on glass. The lights produce two images reflected from the surface, of which the distance apart is observed on the small scale with the telescope. If then

l = the distance of the images from each other;
L = the actual distance of the lights from each other;
A = the distance of the point midway between the lights from the surface;

the radius of curvature r of the surface is—

$$r = \frac{2Al}{L - 2l} \text{ for a convex; or}$$

$$r = \frac{2Al}{L + 2l} \text{ for a concave surface.}$$

The less the curvature, the greater must be the distance A, in order that the formulæ may hold good. On another account also the distance should not be too small, viz., because the

images would not then be sharply visible at the same time as the scale. This objection may be lessened by stopping down the aperture of the object-glass of the telescope.

For lights, the flat flames of petroleum or gas lamps are very convenient if the edge be turned to the reflecting surface. With but little error we may employ the bars of a window, in front of which the observer is placed with the telescope.

Helmholtz's ophthalmometer (18B) is more accurate and more convenient than scale and telescope.

When the curvature of lenses is determined after this manner, there are also images reflected from the further side. In the case of biconvex or biconcave lenses, it is easily seen which are the images to be employed by their erect or inverted position. These images disappear on blackening the hinder surface.

Proof for the above Formula for a Convex Surface.—The distance a at which the virtual image of the line L joining the lights is formed behind the surface is given by $1/a = 1/A + 2/r$, whence $a = Ar/(2A + r)$. On the other hand, the length λ of the image is obviously given by $\lambda : L = a : A$, therefore $\lambda = La/A$.

The image λ appears projected on the scale, with the length $l = \lambda A/(A + a) = La/(A + a)$, whence by substituting the above value of a

$$l = \tfrac{1}{2}rL/(A + r) \qquad \text{or } r = 2Al/(L - 2l).$$

For concave surfaces in a similar manner.

III. *Surfaces of Slight Curvature.*

A telescope is focussed so that an object (say a divided scale) at the distance A is clearly visible. With the telescope thus focussed, let the image of an object reflected from the mirror be clearly seen when the distance between object and mirror is a, that between the mirror and the object-glass of the telescope e. The radius of curvature r is then found by

$$r = 2a\frac{A - e}{A - e - a}$$

It is well to have e about $= A/2$.

If r is positive it denotes a concave surface, negative a convex. The absence of parallax between the cross-wires of the telescope and the image is the test of distinct vision.

IV. *From the Focal Length.*

With slight alterations the focal length of a concave mirror may also be obtained by **44**, 1 and 3–6; that of a convex one by **44**, 9. The radius of curvature is equal to double the focal length.

V. *Testing Plane Surfaces.*

See the method **39**, 8.

A telescope adjusted for a very great distance may also be used to observe the image of a distant object reflected in the surface when this is held close in front of the object-glass of the telescope. The image must show no parallax as compared with the cross-wires. With some practice the naked eye will give fairly sharp results by this method.

A small plane mirror may also be tested by throwing with it an image of the sun upon a distant wall. The image must be round, and of the apparent angular diameter of the sun, taking the mirror as the centre.

If a truly plane glass surface is available the surface to be tested is laid upon it and illuminated with sodium light. The interference bands which appear must be straight and parallel.

44.—The Focal Length of a Lens.

The focus of a lens is the point at which rays parallel to the axis on incidence cross after emergence. The distance of the focus from the lens is the focal length. In concave lenses the focal length has the negative sign. The "number" of a spectacle lens is its focal length expressed in inches.

The "strength" of a lens is determined by the reciprocal of its focal length; a lens or combination of lenses which has the focal length f meters, is said to have a strength of $1/f$ diopters. The strength of a system of lenses placed one behind the other is the sum (having regard to the signs) of

the separate strengths if the thickness of the combination is small compared with the focal length.

The two radii of curvature, r and r' of a lens, and the focal length, are related to each other and the refractive index μ of the glass as follows :—

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} + \frac{1}{r'}\right) \qquad \text{or } \mu = \frac{1}{f}\,\frac{rr'}{r + r'} + 1$$

When a surface is concave its radius of curvature must be taken as negative.

The focal length is different for different colours, and must therefore, strictly speaking, be defined for a special colour (sodium flame or red glass).

In all experiments the axis of the lens (the line joining the two centres of curvature) is brought into the line joining the image and object, as otherwise the distance will be found too small.

(1.) *With the Sun.*—The focal length of a convex lens may be measured by forming with it an image of the sun on a plate of ground glass, and holding it at such a distance that the image is sharp and clear. The distance of the plate from the lens is the focal length.

(2.) *With Telescope.*—A telescope is focussed on some very distant object. Now placing the lens in front of the object glass of the telescope, and looking through it at some plane object (*e.g.* a sheet of paper with writing on it), this will, at a certain distance from the lens, be clearly visible. The distance thus found is the required focal length.

(3.) *From the Distance between Object and Image.*—On one side of the lens is placed a light, and on the other a white screen at such a distance that a clear image of the light is formed upon it. In order to recognise the central adjustment the reflected images of the object in the two refracting surfaces are used. Looking at the lens from the side of the object these images must lie in one plane with the eye and the object.

Taking a and b as the distances of the light and the image from the lens, and f the required focal length—

$$\frac{1}{f} = \frac{1}{a} + \frac{1}{b}, \qquad \text{or } f = \frac{ab}{a + b}$$

(4.) *By Difference of Position of the Lens.*—If an object is at a constant considerable distance l from a screen, there are *two* positions between them which a lens may occupy when forming a distinct image. If the distance between these two positions be e the focal length of the lens is

$$f = \tfrac{1}{4}\left(l - \frac{e^2}{l}\right)$$

As the object we may use cross-wires and instead of the screen another cross, using a lens and determining the coincidence of this and the image by the absence of parallax.

This method (Bessel, *cf.* Oudemans, *Wied. Beibl.* 1879, p. 183) has the advantage that the displacement e can be more accurately measured than the distance from the lens.

Proof.—The distances of object and image from the lens in this experiment are plainly $\frac{1}{2}(l+e)$ and $\frac{1}{2}(l-e)$. Hence $1/f = 2/(l-e) + 2/(l+e) = 4l/(l^2 - e^2)$, *q.e.d.*

(5.) *From Equality of Object and Image.*—When the size of the image is equal to that of the object, the distance between them is equal to four times the focal length.

Thicker Lenses or Systems of Lenses.—The methods given above assume that the thickness of the lens is so small in proportion to the focal length that it may be neglected. When this is not the case, we understand by the focal length the distance of the point of convergence of parallel rays from the *principal plane* of the lens or system of lenses. The principal plane may be found by construction, by producing the directions of incidence and emergence of a ray falling on the lens parallel to its axis till they cut each other, and drawing a plane through the intersection perpendicular to the axis of the lens. The point of intersection of the axis and the principal plane is called the principal point. In lenses of ordinary glass ($\mu = 1{\cdot}5$) of which the radius of curvature is large compared with the thickness d, the two principal points are at a distance from each other of $\frac{1}{3}d$, and further, if both surfaces are equally concave or convex, each is at a distance of $\frac{1}{3}d$ from the surfaces. A plano-convex or plano-concave lens has one of its principal points in the curved surface and the other at $\frac{1}{3}d$ from it within the lens.

The following methods give the true focal lengths of lenses or combinations reckoned from the corresponding principal points:—

(6.) *From the Sizes of much Magnified or Diminished Images.*—On one side of the lens a brightly-illuminated scale is placed, a little farther from it than the focal length (the scale is best of glass with transmitted light). On the other side of the lens a white screen is placed at such a distance that a clear and greatly magnified image of the scale appears upon it. Then taking

l, the length of a division of the scale;
L, the length of its image;
A, the distance of the screen from the lens;

the required focal length f is

$$f = A\frac{l}{L + l}$$

Conversely, also, we may place a sharply-defined object at a great distance from the lens, and measure its image, now much diminished on the other side of the lens. For this purpose it is best to use a scale divided on glass, read by means of a lens which must be so placed that the divisions on the glass and the image of the object are clearly visible through the lens. We must then take, in the previous formula, l for the length of the image, and L for that of the object, and A for the distance of the latter from the lens.

Proof.—The distances A and a of the image and the object from the principal plane of the lens are connected by the formula, $\frac{1}{A} + \frac{1}{a} = \frac{1}{f}$. Their magnitudes are in the ratio $\frac{L}{l} = \frac{A}{a}$. By putting $\frac{1}{a} = \frac{L}{Al}$ in the first equation we get the expression as above. Since A is large compared with the thickness of the lens, we may, instead of the unknown distance from the principal plane, use the distance from the lens.

(7.) *Meyerstein's Method.*—(*a.*) For lenses of long focus, *e.g.* object-glasses of telescopes, let two small millimeter scales divided on glass be fixed on a long wooden stand at a distance c decidedly greater than four times the focal length. The engraved sides of the scales are turned towards each other.

Let now the lens be moved about between the scales until the (diminished) image of the one falls in the plane of the

other, which is recognised by the disappearance of the parallax, and the ratio of image to object be determined $= v$; let also the distance from the object of some point firmly connected with the lens, *e.g.* some point on the edge of the cell, be measured $= a$.

Let now the lens be reversed and the operation repeated; the agreement of the two values of v shows the accuracy of the adjustments. Let the distance of the point connected with the lens from the object be now b' and we have for the focal length

$$f = \frac{a + b' - c}{1/v - v}$$

(*b.*) In the case of the eyepieces of telescopes and microscopes, and the objectives of microscopes, the image is too small to be measured as above; in addition, the position of the image often falls in the interior of the combination.

We may here with advantage use a microscope of low power with an eyepiece micrometer and laid horizontally, and, as the object, rectangular blocks of suitable size or bits of coloured paper on a white ground. We obtain v at once from the size of the object, and the size of the image measured by the micrometer divisions of known value. a and b' are easily measured direct. In order to obtain c the lens is removed and the distance from the object to a needle-point placed in the position where it is in good focus, measured.

Cf. Meyerstein, *Wied. Ann.* i. 315, 1877.

(8.) *Abbe's Method.*—Let the ratio v_1 of magnified image to object be determined for some position of the lens, etc.; let the object be moved by a measured amount Δ and the ratio v_2 found. The focal length is

$$f = \frac{\Delta}{\frac{1}{v_2} - \frac{1}{v_1}}$$

The apparatus is the same as for Meyerstein's method. The method has the advantage that no determination of the *place* of the image is necessary.

Abbe has constructed a special apparatus (focometer) for the convenient performance of the measurement.

(9.) *Diverging Lenses.*—A concave lens, which gives no real image, is used in combination with a stronger convex lens of known focal length, and the common focus of the two determined by one of the methods (1) to (4). If, then,

F = the common focal length;
F' = that of the convex lens alone;

the focal length f of the concave lens will be found by

$$\frac{1}{f}=\frac{1}{F}-\frac{1}{F'}, \text{ or } f=\frac{FF'}{F'-F}$$

(10.) Finally, the focal length of a concave lens may also be obtained by measuring the circle of light formed by the diverging rays of the sun when thrown on a screen at a given distance. If

d = the diameter of the aperture of the lens;
D = the diameter of the circle of light;
A = the distance of the screen from the lens;

we have for the focal length—

$$f=\frac{Ad}{d-D+0{\cdot}0094A};$$

0·0094 is twice the tangent of the apparent diameter of the sun. When the lens is deep and not too small, this term may be neglected, and we thence obtain the simple rule, to take for the focal length that distance at which the circle of light on the screen is double the aperture of the lens.

(11.) *Focal Length of Weak Lenses* (*cf.* **43**, III.).—Let a telescope be focussed for distinct vision of an object at the distance A from the object-glass. The lens is now introduced close in front of the object-glass, and the distance of the object varied until it again appears distinctly focussed. Let this be the case for a distance A'. The focal length of the lens is then

$$f=\frac{AA'}{A-A'}.$$

Negative focal length denotes a diverging lens. Distinct vision is recognised by the absence of parallax between the image and the cross-wires.

[*Thompson's Focometer.*—S. P. Thompson describes (*Jour. of Soc. Arts,* xl. p. 22) another method of determining not only the focus but also the positions of the principal points of any lens, simple or compound, thick or thin; and an instrument designed to facilitate the measurements.

The method depends upon the fact that the conjugate foci with equality of object and image, the *symmetric points* as he calls them, are as far from the *principal* focus on each side as this is from its corresponding *principal point.*

The lens is fixed on the instrument between two glass micrometer scales carried by supports which slide independently on the longitudinal graduated frame of the instrument, but are also, when suitably clamped, moved equally in opposite directions by the right and left handed parts of a main screw, or for less accurate work by an arrangement of the nature of "lazy-tongs," as suggested by Everett, or by clamping the two scales to the two parts of a cord passing round a pulley at one end of the apparatus, as originally used by Thompson.

The one micrometer is placed in the principal focus of the lens, which is found by using as an object a set of black lines on glass placed in the focus of a collimating lens, and moving the micrometer until there is no parallax between its divisions and the image of the black lines. The reading on the graduated beam is then taken, say F_1.

The object and collimator are then placed at the other side of the lens, the first micrometer turned out of the way by a hinge arranged for the purpose and the position of the principal focus in the other direction determined in the same manner F_2. By means of a plumb line the position of some point of the lens, *e.g.* one surface of it, is found to be at A on the scale.

The two micrometer supports are then clamped to their respective ends of the screw, and turning this so as to make the micrometers travel outwards, a position is reached at which the image of one micrometer exactly coincides with the other one. Call the readings S_1 and S_2.

f, the focal length, is $F_2 - S_2$ or $S_1 - F_1$ so that we may take the mean.

K, the distance between the principal points is

$$2(F_1 - F_2) - (S_1 - S_2).$$

The positions of the principal points on the graduation are $2F_1 - S_1$ and $2F_2 - S_2$, and their distance from the point of the lens determined (at A) can be thus found.] (*Tr.*)

45.—Magnifying Power of an Optical Instrument.

I. *Lens.*

The magnifying power of a lens is calculated from the focal length, which, for a thick lens or a combination of lenses, must be determined by (6) of the previous article. Calling

f = the focal length;
A = the least distance of distinct vision with the naked eye;

the magnifying power of the lens is—

$$m = \frac{A}{f} + 1$$

For ordinary eyes the least distance of distinct vision may be taken as 25 cm.

Proof.—If a small object of the length l be placed at a distance a under the lens, so that its (virtual) image appears at the distance A, we have $\frac{1}{a} = \frac{1}{A} + \frac{1}{f}$. Let the image have the length L, and the magnification will be $\frac{L}{l} = \frac{A}{a} = 1 + \frac{A}{f}$.

II. *Telescope.*

The magnifying power is the ratio of the angle which a distant object subtends, when seen through the telescope, compared with that which it subtends as seen with the naked eye.

(1.) The following method is universally applicable for determining the magnifying power:—The telescope is placed at a distance, which must be great compared with the length of the instrument, before a measuring rod (a paper scale, a slated

roof, or the pattern of a wall-paper will answer the purpose), on which two points must be sufficiently marked to be seen with the naked eye. Looking now at the scale through the telescope with *one* eye, and with the *other* unassisted, the two images are seen superposed. If, then, the distance between the two points appear to correspond to n divisions of the scale, as seen through the telescope, while the actual distance is N divisions, the magnifying power is—

$$m = \frac{N}{n}$$

The observation will be much facilitated by drawing out the eyepiece of the telescope, so that the two images are not displaced relatively to each other by a movement of the axis of the eye. A short-sighted eye must, of course, be assisted by spectacles.

(2.) At short distances, the following method (Waltenhofen's) may be used:—The telescope is focussed for a very distant object, and then a thin convex lens of low power (spectacle glass of about 2 m. focal length) is fixed in front of its object-glass. The telescope is then pointed to a scale at such a distance that the divisions appear well defined. Just as in No. 1, an observation is made with both eyes. If n divisions, as seen in the telescope, coincide with N as seen with the naked eye, and if the distance of the scale from the object-glass be a, and from the eye A, the magnifying power of the telescope is—

$$\frac{N}{n} \cdot \frac{a}{A}$$

(3.) In telescopes with convex eyepieces the following simple method is almost always applicable:—First, the telescope must be focussed for an object at infinite distance. The object-glass is then taken out and replaced by a cardboard screen with a rectangular opening, or by a transparent scale. The remaining lenses of the telescope will form a real image of the screen or scale which is measured on a small glass scale placed in front of the eyepiece, using a lens for observing. The size of the image divided into the actual size gives the desired magnifying power.

The circular opening of the cell of the object-glass may itself be used for the above-mentioned screen if we are sure that the rays coming from its edges are not cut off by the diaphragms of the tube, as is frequently the case. A screen of angular form shows at once if this is the case.

Proof for Kepler's Telescope.—If F be the focal length of the object-glass, f that of the eyepiece, the magnifying power is, as is well known, F/f. The distance of the eyepiece from the objective is, when a distant object is distinctly seen, $A=F+f$. The object of the length L in the place of the object-glass gives therefore an image of length $l=fL/(A-f)=fL/F$ (see previous article, No. 6). Therefore $L/l=F/f$.

(4.) The following method (Gauss) is more accurate, depending on the principle that a telescope focussed for an infinitely distant object gives, when the rays pass through it in a reversed path, a reduction of size which is equal to the magnifying in the ordinary use of the instrument. The angular magnitude of a distant object is measured by means of a theodolite, first directly, then through the telescope placed, reversed, in front of the theodolite. The quotient of the two angles gives immediately the astronomical magnifying power.

In a limited space the method may be carried out modified as follows:—A horizontal bar with two (or better, several) marks on it symmetrically placed with regard to the centre is placed at a distance of at least 1 meter in front of the eyepiece of a telescope focussed for a distant object. Let the distance of these marks apart be a. In the theodolite placed in front of the object-glass let these marks appear under the angle ω. The corresponding angle at which the rays enter the telescope ϕ is given by the fact that they must pass through the diaphragm of the eyepiece. If A be the distance of this from the bar we have

$$\tan \tfrac{1}{2}\phi = \tfrac{1}{2}a/A$$

and the magnifying power of the telescope $=\phi/\omega$.

Instead of a theodolite a spectrometer telescope may be used, or still better, a reading telescope, moved by a micrometer screw, and with a mirror attached to it, reflected in which a scale is read off by means of a second telescope.

(5.) The focal lengths of the separate lenses and their distances being known, the power may be calculated. For example, that of an astronomical telescope with a simple eyepiece, or of a Galilean telescope, is the focal length of the objective, divided by that of the eyepiece. Practically, the results of these and similar rules are of little use, since the focal length of the Galilean eyepiece cannot be measured directly, and eyepieces with convex lenses are mostly of a compound nature. The exact measurement of the distances, frequently very small, between the lenses in the eyepiece offers great difficulties; and besides this, without determining the position of the principal points, only a rough result can be obtained from the formulæ.

(6.) *The Size of the Field of View.*—If l be the actual distance from each other of two points at the ends of a diameter of the field of view and a their distance from the telescope, the size of the field of view expressed in degrees is $57^{\circ}{\cdot}3 \,.\, l/a$.

In practice a measuring-rod fixed at some distance is again the most convenient object to employ. If a great distance is not available, a weak lens in front of the object-glass may be used as in No. 2, and the scale brought to the distance for distinct vision; a is then the distance of the scale from the lens.

III. *Microscope.*

(1.) The magnifying power of a microscope may be taken as the ratio of the angle under which a small object is seen in the microscope, to that which the same object would subtend at the distance of 25 cm.

The method of determining the power of a microscope is the same as that described in No. II. (1). An object (stage micrometer) the length of which is known, is brought under the microscope. At 25 cm. below the eyepiece is fastened a rule. Whilst one eye sees the object through the microscope the other sees the rule, and measures upon it the projection of the image visible in the microscope. If the image appear N divisions in length, whilst its actual size is n divisions, the power is N/n.

Better still, a small mirror, the silvering of which has been rubbed off in the middle, may be fixed over the eyepiece at an

angle of 45°, and the scale set up 25 cm. to one side, so that with the eye the same object under the microscope and the reflected image of the scale are both visible at once.

Instead of comparing the image of the object with a scale at the distance of 25 cm. it may be drawn upon a surface at that distance from the eye and the drawing measured.

(2.) Upon the measurement of lengths with the microscope see **18**, 3.

(3.) *Angle of Aperture and Numerical Aperture of a Microscope Objective.*—The angle of aperture ($2u$) is the angle between the extreme rays which, proceeding from a point distinctly seen in the axis of the instrument, can pass through the microscope.

Let μ be the index of refraction of the medium from which the rays enter the objective. Then $a = \mu \sin u$ is called the numerical aperture of the objective. In dry combinations $\mu = 1$ and therefore $a < 1$. In immersion lenses μ depends on the fluid used (water, oil of cedarwood, or monobromnaphthalin), and here we may have $a > 1$.

Upon the numerical aperture depends the so-called resolving power of the microscope, *i.e.* the size of the smallest object distinguishable by it.

The lines of a grating distant ϵ will be separated using *central* light when $\epsilon \geqq \lambda/a$, with suitable oblique illumination when $\epsilon \geqq \lambda/2a$ where λ is the wave-length of the light in air (Abbe).

According to Lister the numerical aperture of a dry objective can be determined by laying the microscope horizontal, and moving before it, in the dark, a flame until one half of the field of view appears bright, and in the same plane perpendicular to the axis placing another flame in such a position that the other half is also illuminated. If the distance of the flames be e and the plane in which they are be distant A from the point of distinct vision, we have $\tan u = e/2A$.

Apertures above 1 are measured by the apertometer of Abbe, which is always available. On a semicylinder of glass of 9 cm. diameter and 1·2 cm. in height a reflecting prism of 45° is ground on the edge which forms the diameter, and the centre is marked by a little aperture in a silvered cover-glass cemented on to it.

The upper surface has two graduations corresponding to the angle of aperture and the numerical aperture; two blackened brass plates bent at right angles serve as indices.

The apertometer is placed upon the stage of the microscope which is focussed (if an immersion lens is used the proper fluid being interposed) to distinct vision of the little aperture. Taking away the eyepiece and looking down the tube we see with low powers the image formed by the objective of the two indices, the points of which are adjusted to the edges of the field of view and the division read.

With higher powers the images are too small to be observed with the naked eye, and an auxiliary microscope is used, which in Zeiss's instruments is formed by screwing a special low power objective into the lower end of the draw-tube and using an eyepiece. In other instruments a suitable objective may be fixed into the draw-tube with cork.

45A.—Angle of Polarisation of a Body.

The light reflected from a mirror is completely polarised at that angle of incidence or reflection at which the path of the ray which enters the body is at right angles to that which is reflected. Hence it follows, calling this "polarisation" angle ω and the index of refraction μ,

$$\mu = \tan \omega$$

If μ is known, ω can be calculated from this, or μ can be found when ω is given.

In order to measure ω directly a Nicol's prism is used, the polarisation direction of which (the longer diagonal of the face) lies perpendicular to the plane of incidence of the light. The mirror is illuminated by a broad source of light, such as translucent paper before a sodium flame, and the reflected light is observed through the Nicol's prism. At the right position an ill-defined dark band appears in the field of view; the line of vision corresponding with the middle of this makes the polarising angle with the normal to the surface which can be determined as follows with the spectrometer (**39**). The Nicol is fixed (in accurate measurements, in conjunction with a

telescope), and the mirror fixed to the spectrometer axis and provided with a mark in the axis, is turned from one minimum of brightness to the other upon the opposite side, of course moving the source of light. The movement has amounted to 2ω.

In the case of the surfaces of fluids 2ω is determined by setting some vertical rotating observing instrument (Wollaston's goniometer, theodolite, spectrometer, turned over) on the dark spot from both sides.

46.—Optical Rotating Power. Saccharimetry (Biot).

If the field of view of a polariscope with crossed Nicol prisms become light when a transparent body is inserted, this is either doubly refracting or "optically active," *i.e.* it rotates the plane of vibration of the polarised light. A substance is said to have "right-handed" rotation when the plane of vibration is displaced in the opposite direction to the spiral of a corkscrew, *i.e.* when, to the eye receiving it, it appears turned in the direction of the motion of watch-hands.

The angle of rotation produced by the unit of length of the active substance is called the specific rotation. If a solution of an active substance in a neutral medium be in question this angle of rotation must be divided by the amount of the substance in unit volume of the solution. The specific rotation multiplied by the molecular weight of the substance is called the molecular rotation.

The rotation depends greatly upon the colour of the light. The amount of rotation is greatest in the case of the most refrangible light. Red light for the investigation is obtained by a glass coloured with suboxide of copper held before a white flame or by the lithium flame, yellow by the sodium flame, green by the thallium. Many green glasses also, or stained gelatine films, give fairly homogeneous light. On observations with light of a spectrum see the end of this section.

Solutions of sugar are those most frequently observed for rotating power. We shall confine ourselves to the instruments which are used for this purpose. The rotation of other bodies can be measured in exactly the same way.

The specific rotation of cane sugar dissolved in water is, for the yellow sodium flame, $66^\circ \cdot 5$/dm., *i.e.* the angle of rotation a produced by a solution containing in 100 c.c. z grm. of sugar is for a length of solution of l dm.

$$a = 0^\circ \cdot 665 \,.\, zl, \text{ whence } z = 1 \cdot 504 \frac{a}{l}$$

For white light (average)

$$a = 0^\circ \cdot 71 zl, \text{ whence } z = 1 \cdot 41 \frac{a}{l}$$

Strictly speaking, the rotation increases a little faster than the amount of sugar. For sodium light a more accurate formula is (Schmitz)

$$z = 1 \cdot 501 \frac{a}{l} + 0 \cdot 00031 \left(\frac{a}{l}\right)^2$$

The specific rotation of quartz in the direction of the axis is for sodium light $21^\circ \cdot 7$/mm.

Taking the rotation for the yellow sodium light as unity, the rotations for other colours are almost exactly in the same ratio in the case of both quartz and sugar, on the average about as follows:

Middle	Red	Yellow	Green	Blue	Violet
Rotation	3/4	1	4/3	5/3	9/4

From this, with the help of the numbers already given, the appearance of the colouring can be deduced.

The rotative power d can be expressed as a function of the wave-length λ (Table 19A) by the formula $d = a/\lambda^2 + b/\lambda^4$.

Accurate particulars for quartz are given in Table 20; for other bodies *cf.* Landolt, *Optisches Drehungsvermögen*, 1879; Landolt u. Börnstein, *Tabellen*, pp. 97-101.

The instruments for measuring the rotation (saccharimeters) have either a divided circle on the polarising arrangement, by the rotation of which the rotatory power of the substance is measured (Mitscherlich), or a "compensator" (quartz wedge), the pushing in of which answers the same purpose (Soleil).

I. *Saccharimeter with Rotating Nicol.*

(1.) The original instrument of Mitscherlich consisted solely of a fixed polarising prism P and an ocular prism A

rotating over a divided circle *K*. A soda flame (Bunsen's gas-flame with soda bead on a platinum wire) is placed behind the instrument in front of a black screen. The blue light of the Bunsen's flame itself may be suitably absorbed by a yellow glass or a cell containing solution of bichromate of potash. Then a tube, empty or filled with pure water, is placed between the Nicol's prisms of the instrument, and the Nicol in the eyepiece turned until the middle of the field of view appears dark. Finally the tube is filled with the solution of sugar (very thoroughly mixed) and put into its place again. The field of view, with the first position of the index, appears bright. The number of degrees through which the index must be turned to the right (in the direction of the hands of a watch), that the centre of the field may be dark again, is the required angle of rotation *a*.

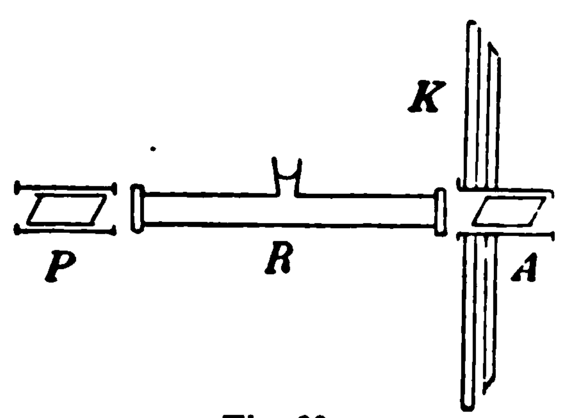


Fig. 29.

If we intend the zero of the circle to be also the point from which the angle is measured, the index is put to zero without any sugar solution, and the farther Nicol turned until the centre of the field is dark.

For many eyes the use of a weak lens or spectacles in front of the eyepiece is an advantage.

The angle of rotation of a solid body, *e.g.* a plate of quartz cut perpendicular to the axis, is measured exactly as above by introducing the body between the two polarising prisms. The quartz plate must be placed exactly at right angles to the axis of the instrument if very deceptive appearances are to be avoided. The plate may be adjusted by observing the reflected image of the eye or of a small flame held just in front of it.

When white light is used, since the colours are rotated differently, there is no longer after the introduction of the rotating solution any position in which the field is dark, but a change of colour follows the rotation of the prism in the eyepiece. The adjustment is made for the "sensitive tint" in which the yellow is extinguished, *i.e.* a violet passing rather abruptly into red on one side and blue on the other. For the calculation the constant is 1·41 (p. 191).

The question as to whether the rotation is "left" or "right" handed is decided by the fact that it is whichever direction the eyepiece Nicol must be turned through to produce the "sensitive" change from blue to red. Finally, should there be a doubt as to whether the angle of rotation is greater or smaller than 180°, two observations are made, one with red light (ruby glass) and one with the soda flame. The two rotations are approximately in the ratio, yellow : red = 4 : 3.

A greater sharpness of the adjustment is attained by the following modifications of Mitscherlich's instrument.

(2.) *Double Quartz Plate.*—Two quartz plates of equal thickness, most suitably 3·75 mm., cut from right and left handed crystals respectively, are placed side by side in front of the polarising prism, accurately perpendicular to the line of sight.

Whether the Nicols are crossed or parallel, both plates appear, with the soda flame equally bright, with white light equally coloured. Plates of 3·75 mm. in thickness produce with parallel Nicols the so-called "sensitive tint," and are also very sensitive in sodium light, which they rotate through about 80°.

When a rotating substance has been introduced the two halves become dissimilar. The eyepiece Nicol is now turned through the rotation angle α of the substance, and equality is again established. If the rotation of the substance is considerable the dispersion of the colours of the white light thereby produced prevents an exact equality of the halves of the plate. In this case it is better to conduct the observations with monochromatic light.

(3.) *Polaristrobometer* (Wild).—This instrument gives by means of a Savart's plate (two quartz or calcite plates cut at an angle of 45° to the axis with their principal sections at right angles), a field of view with bands which in homogeneous light are bright and dark, in white light coloured. The eyepiece is first so far pulled out that the bands appear as sharp as possible. The adjustment for saccharimetry is for the disappearance of the bands in the middle of the field of view. Since it is the Nicol's prism furthest from the eye which is rotated, the rotation as seen from the eyepiece end

must be taken in the direction contrary to that of the hands of a watch. The bands disappear in four positions 90° from each other. As to the question whether the angle of rotation a is greater or less than 90°, see section (1).

The instruments frequently have a second graduation which when a tube 200 mm. long is used gives directly the sugar in 1 liter of solution.

(4.) *Half-Shadow Apparatus of Laurent.*—Sodium light is used. Half of the field of view is covered by a plate of crystal (mica or quartz cut parallel to the axis) which displaces the two component rays into which the ray is divided on its passage through the plate (fig., a) by half a wave-length with respect to each other. On emergence the components unite again to a single wave the plane of vibration of which is therefore rotated with respect to that of the incident ray (fig., b), so that rays emerge from the covered and the uncovered halves with different directions of vibration, as in the case of the double quartz. The zero is that position of the eyepiece with which the two halves appear equally bright. The sensitiveness of the change of brightness depends upon the position of the plate of crystal with regard to the polariser. The best position for the strength of light used is found by experiment, and the instrument fixed at this. The zero is then determined.

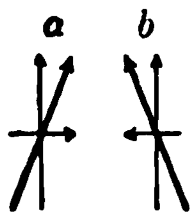


Fig. 30.

When a rotating substance is introduced the analyser must be turned back through the rotation angle, in order to again produce equal brightness in the halves of the field.

(5.) *Half-Shadow Apparatus of Lippich.*—The light passes first through a large polarising prism with perpendicular faces (Glan's prism), then through another similar one which only takes in half the field of view. The first prism can be rotated round its axis by means of a lever, so that the angle between the two planes of polarisation can be varied.

The observation requires homogeneous light, which, however, may have any wave-length.

The instrument is adjusted to equality of brightness in the two halves of the field, with and without the intervention of the active substance; the angle between the two readings gives at once the amount of the rotation.

If the substance is clear the angle between the planes of polarisation in the polariser is made small since the adjustment is then sharper; with cloudy substances the angle must be larger in order to obtain enough light. Finally, the zero must be found, which of course depends on the position of the Nicol's prism.

(6.) *A Cornu-Jellett's prism* also gives two halves of the field of view which must be adjusted to equal brightness.

II. *Saccharimeter with Quartz Wedge* (Soleil).

The rotation of the plane of polarisation by a solution of sugar may be compensated by a plate of quartz of opposite rotation, and this not only for monochromatic but also for any light, since the dispersion of the colours by quartz is very nearly proportional to that by a solution of sugar. Wedge-shaped quartz plates, of which any desired thickness can be introduced, permit the rotation in the sugar to be deduced from the thickness necessary to compensate the rotation.

Description of Soleil's Saccharimeter.—The light enters through the polarising Nicol *P*, and passes thence through the double quartz plate *D* (see I., 2). Then follows the tube *R*, which can be filled with the solution. Then the compensator, consisting of a plate of

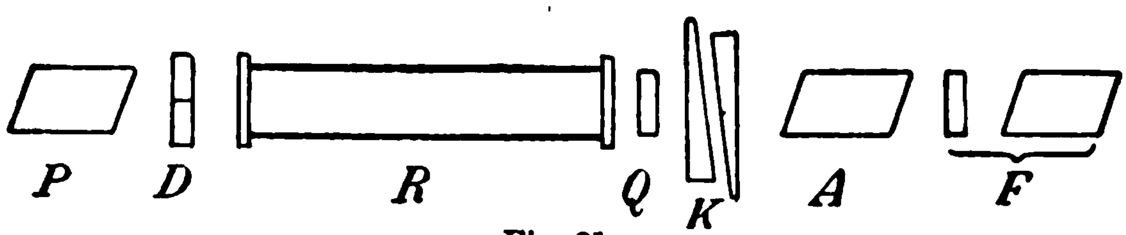


Fig. 31.

right-handed quartz *Q* and the two wedges of left-handed quartz *K*, which can be pushed over each other by a rack and pinion, thus forming a left-handed plate of variable thickness. In the mean position the resultant thickness is the same as that of the right-handed quartz *Q*, so that *Q* and *K* together are without any action. This position should correspond with the zero of the graduation connected with the rackwork. After these comes the analysing Nicol *A* having its plane of polarisation parallel to that of *P*.

Since the solutions of sugar, etc., may be coloured, and since also all eyes are not equally sensitive to the same changes of colour, the violet transition tint is not always the most delicate. On this account there is, as a rule, at the eye-end (in many instruments on the other hand in front of the flame) a colour-regulator *F*, consist-

ing of still another quartz plate and Nicol's prism, by the rotation of which the colour of the field of view is altered. This rotation has of course no effect on the zero of the instrument.

The tube, empty, or filled simply with water, is put in position, a white lamp flame or daylight used as the source of light, and the small telescope connected with the eyepiece (not shown above) drawn out until the quartz plates appear sharply bounded. In order to obtain the most suitable tint the semicircles are brought by means of the rack to not quite the same colour. By turning the colour-regulator (see above) that tint is obtained which gives the greatest difference of the parts of the field.

By sliding the wedges the colours are made the same; the graduation is then read, the solution of sugar introduced, the adjustment again made and read off, both readings being several times repeated.

In the instruments with quartz compensation, which are most used, the motion through one division corresponds to a revolution of the yellow sodium light—

in the Paris instruments (Soleil Duboscq) of $0^\circ{\cdot}217$
in the Berlin instruments (Soleil Ventzke) of $0^\circ{\cdot}346$

The sugar contained in 100 c.c. of the solution in grammes, will therefore be, using a tube 200 mm. long, when the displacement from the position when the tube is empty is p divisions—

Soleil Duboscq $z = 0{\cdot}1635\,.\,p$
Soleil Ventzke $z = 0{\cdot}2605\,.\,p$

For specimens of sugar, therefore, in which the percentage of pure sugar is to be determined, the rule is: dissolve 16·35 (or 26·05) grms. of the sugar to 100 c.c. of the solution; the displacement then gives the percentage of pure sugar.

To test the accuracy of the divisions, a "normal solution" of pure sugar containing 16·35 (or 26·05) grms. in 100 c.c. is used. The displacement must amount to 100 divisions. Divisions of unknown value are determined by experiments on solutions of known strength, or on plates of quartz.

If we wish the zero of the divisions to coincide with no sugar in the solution, the index is placed at the zero, when the empty tube is in its place, and the polarising prism rotated until the semicircles have the same colour.

Determination of Sugar in the presence of other optically active Substances.—The elimination of the influence of other optically active substances, besides cane-sugar (*e.g.* inverted sugar or dextrin), depends upon the fact that the cane-sugar, rotating the plane of polarisation to the right, is, by warming for ten minutes to about 70° with hydrochloric acid, changed into inverted sugar, which has left-handed rotatory power. Whilst the rotatory power of solutions of cane-sugar is independent of the temperature, that of solutions of inverted sugar varies rather considerably with changes of temperature. An inverted solution l mm. long, which contains in 1 c.c. z grms. of what was originally cane-sugar, rotates the plane of polarisation of the sodium flame, at the temperature t', through the angle (Tuchschmid)

$$(0^\circ{\cdot}2933 - 0^\circ{\cdot}00336t')\, zl$$

Hence the practical rule:—After the rotation (*i.e.* the angle a or the displacement p of the quartz wedge) has been determined with the usual solution, 100 c.c. of the solution are taken, mixed with 10 c.c. of strong hydrochloric acid, and kept for 10 minutes at a temperature of 70°. When the fluid has cooled, a tube one-tenth longer than the former one is filled with it (or if the same tube be used, the angle obtained must be multiplied by 1·1), and the rotation to the left a' (or p'), which now is produced, is observed. Let the temperature of the solution at this latter observation be t'. The rotation due to the cane-sugar alone is then

$$\frac{a + a' \text{ (or } p + p')}{1{\cdot}442 - 0{\cdot}00505t'}$$

For if the rotation due to substances not sugar which we wish to eliminate is called $= \beta$, we have (p. 191 and above)—

$$a = 0{\cdot}665zl + \beta$$
$$a' = (0{\cdot}2933 - 0{\cdot}00336t')\, zl - \beta$$

Consequently—

$$a + a' = (0{\cdot}9583 - 0{\cdot}00336t')\,zl$$
$$= (1{\cdot}442 - 0{\cdot}00505t')\,.\,0{\cdot}665zl$$

But $0{\cdot}665zl$ is the rotation due to the sugar alone.

Determination of Rotation in the Spectrum.—If the polarisation apparatus of Mitscherlich is used with compound light (sunlight), that which has traversed the instrument may be decomposed by means of a spectroscope. The "crossed" position of the Nicols is shown by the whole spectrum becoming dark. The introduction of a rotating substance makes the spectrum bright. If the analyser is turned to follow the rotation, a dark band appears in the spectrum, which on further rotation of the Nicol passes from the red to the violet end. The middle of this band corresponds with that light which is completely extinguished. By the position of the analyser at the time the rotation for this particular colour is therefore measured.

46A.—Investigation of Doubly Refracting Bodies. Recognition of the Optical Characters of Uniaxial Crystals.

A body refracts light either singly or doubly; the former when it is amorphous or crystallised in the cubic or regular system, the latter when it belongs to one of the not regular systems of crystals or when it has received different properties in different directions from other causes, such as pressure, strain, rapid cooling.

Bodies are separated into these two classes by the aid of the polarisation apparatus, *i.e.* a combination of two arrangements which polarise the light. For this purpose may be used Nicol's prisms, tourmaline plates, unsilvered and usually black glass plates from which the light is reflected at an angle of incidence of 56°, or sets of glass plates laid over one another, through which the light passes at the same angle of incidence. Doubly refracting prisms of calcspar or quartz divide the light into two rays vibrating in planes at right angles to each other; the colour separation produced at the same time may be removed by a glass prism cemented on. For many pur-

poses a pencil of light with different directions in the crystal (large field) is required, to produce which convex lenses are inserted between the crystal and the polariser (Norremberg's polarisation microscope). For observations on small bodies in polarised light with the ordinary microscope, a Nicol's prism is introduced between the mirror and the body, and another is placed over the eyepiece of the microscope, or immediately behind the objective.

The polarising arrangement nearest the eye is termed the analyser, the other the polariser simply.

The polarising apparatus is mostly used with the polarising arrangements "*crossed*," when the field of view appears dark. The two planes of polarisation of the arrangements, in this case at right angles to each other, are called the "principal planes" of the apparatus.

Whether a body is singly or doubly refracting is determined with the prisms crossed. A singly refracting substance leaves the field of view dark, with the exception of those few bodies which exert a rotatory power on the light (**46**) without double refraction. A doubly refracting body makes, generally speaking, the field of view bright or coloured. Only in special positions, and then only in a small field of view, does the darkness continue.

If a plane plate of a doubly-refracting crystal be employed, the light divides in its passage through the plate into two series of waves, polarised at right angles to each other. The planes of vibration are easily recognised when the plate is placed between the polarising arrangements. For then the plate has two different positions differing by 90°, at which the field of view, or at any rate the middle of it, remains dark. In these positions the planes of vibration coincide with the principal planes of the apparatus.

Uniaxial Crystals.—One of the two planes of vibration must be a "principal plane," *i.e.* must contain a principal axis. If the middle remains dark throughout the rotation of the plate, it shows that the plate is cut perpendicular to the axis. In an apparatus with large field (tourmaline pincette, polarisation-microscope) the darkness extends from the centre of the field of view in the two principal planes of the apparatus (the

dark cross); the four quadrants are traversed by rings which in monochromatic light (red glass held before the eye) are alternately light and dark, in white light are coloured. Bodies such as quartz which rotate the light do not usually show the black cross in the central part of the field.

The more closely the rings lie together, so much the greater, when plates of equal thickness are compared, is the "double refraction," *i.e.* the difference between the velocities of the ordinary and the extraordinary ray.

Preparation of a Plate of a Crystal.—This is done in the case of a hard crystal with a wire or saw with emery, or by grinding on a stone or on a glass plate with emery, or in the case of softer substances with pumice or simply upon a ground-glass surface. A superficial polish is then given with rouge, tripoli or putty powder, or with very soft bodies without any polishing material upon a glass plate closely covered with linen or paper, in order to observe under the polarisation apparatus whether the plate is correct in direction. If not, the surfaces are corrected until the right direction is obtained and the polish completed after thorough fine grinding, best with straight movements of the crystal, often changing the direction of them. Alcohol may be used in the grinding and polishing if the body is attacked by water. In the last case the finger may be covered with india-rubber. Rouge or tripoli are used dry.

Recognition of Positive and Negative Crystals.—A crystal in which the extraordinary ray is more refracted than the ordinary one is called positive and *vice versâ.*

The sign is recognised by means of a so-called quarter-undulation or circularly-polarising mica film, *i.e.* a film of such a thickness that the two vibrations which traverse the plate experience a difference of phase of a quarter of a wave-length. This mica-film is placed anywhere between the prisms, in such a position that the plane of the optic axes of the mica is at an angle of 45° with the principal planes of the apparatus. The crystal plate being examined then no longer shows the black cross with similar quadrants of rings, but the parts of the rings are displaced relatively to each other in alternate quadrants, and in the neighbourhood of the centre, which is now bright, are two dark spots. If these spots lie in the plane of the optic axes of the mica the crystal is negative and *vice versâ.*

Mica can be easily split to the required thickness. It is

known to be suitable for use, and the direction of its optic axes is determined most simply by using it on a known crystal (*e.g.* calcspar, negative; quartz, positive). The plane of the axes may also be determined from the figure given in converging light (see next page).

The phenomena are explained on Fresnel's hypothesis as follows: Assume that the crystal is negative, that therefore the extraordinary ray, *i.e.* the ray vibrating radially in the apparatus, has a greater velocity than the ordinary ray, with its vibrations tangential. At a certain inclination to the axis (*i.e.* in the interference figure, at a certain distance from the centre which must lie inside the first dark ring), the ray with radial vibrations will have gained on the other by $\frac{1}{4}\lambda$; for the first dark ring corresponds to a difference of phase of $\frac{1}{2}\lambda$.

Now in a plate of mica a ray with vibrations parallel to the plane of the axes is propagated more slowly than in other directions; our quarter-undulation plate therefore retards vibrations in its axial plane a fourth of a wave-length compared with the vibrations of the other component. If now of the above-mentioned rays of which the radial component is accelerated $\frac{1}{4}\lambda$ in the crystal, those are received into the eye which lie in the plane of the axes of the mica, it is seen that the difference of phase is increased by the mica. The two dark spots therefore appear near the centre of the field in the plane of the axes of the mica.

It follows of course that a positive crystal must behave in an exactly opposite manner. Similarly it is easily seen that the diameter of the rings is increased by one-fourth of the distance between them in two quadrants, and in the two others diminished by an equal amount.

On the measurement of refractive indices of crystals compare **40**.

47.—Determination of the Angle of the Optic Axes of a Crystal.

Let a plate be cut from an optically biaxial crystal perpendicular to the bisectrix of the axes. When the prisms of a polarising apparatus are crossed (**46**A), such a plate gives, if the field of view is sufficiently large, a system of bright and dark (or coloured) lemniscates traversed by a black cross or by hyperbolic dark brushes. The vertices of the hyperbolas round which the lemniscates contract denote

the optic axes of the crystal. When the line joining the images of the axes coincides with the plane of polarisation of either polariser or analyser the dark cross appears (*a*). If the crystal plate is rotated through 45° from this position, the

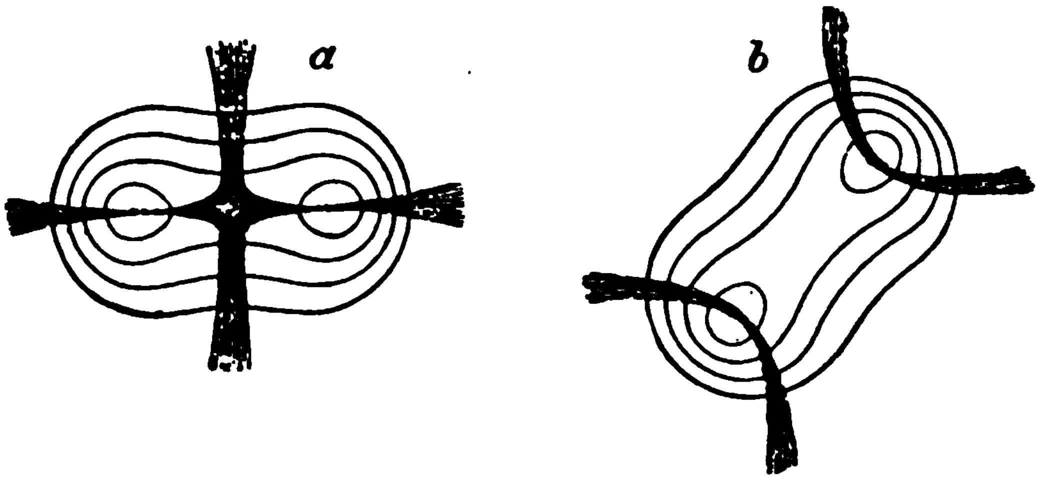


Fig. 32.

dark hyperbolic brushes appear symmetrically disposed relatively to the lemniscates. This appearance depicted in *b* is the most suitable for the measurement of the angle between the axes. A mark is made on the plate perpendicular to the line joining the optic axes.

A small arrangement made of brass consisting of a divided circle on the axis of which the plate of the crystal is fixed with wax or cork, and which can be fixed by means of a ring on the lower part of the Norremberg apparatus, is easily prepared. A special apparatus for determining the angle of the axis is prepared by Fuess after Groth's directions.

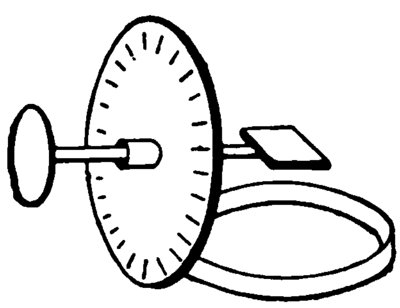


Fig. 33.

In order that the figure, as in *b*, may be obtained, the axis of the circle must make an angle of 45° with the principal planes of the prisms. The plate is now fixed to the axis of the circle so that the marked direction lies in this latter, and one of the optic axes (vertex of the hyperbola) is brought to coincide with the cross-wires in the field of the apparatus. The reading of the circle is then taken. The angle a through which the plate must be rotated in order that the other vertex may fall on the cross-wires, is the *apparent* angle of the optic axes, *i.e.* the angle after their emergence into the air of the rays of light which traverse the crystal in the directions of its optic axes.

If the mean index of refraction μ of the crystal is known

(**40**, II.; Table 20), the *true* angle a_0 of the optic axes in the crystal is found from the expression

$$sin\ \tfrac{1}{2}a_0 = \frac{1}{\mu}\ sin\ \tfrac{1}{2}a$$

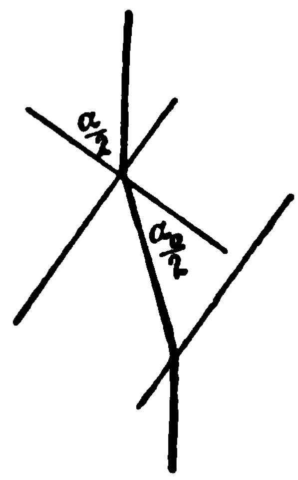


Fig. 34.

In the case of axes with a larger angle between them of course only one axis is visible at a time. If the angle is still greater it may happen that on account of refraction and total reflection, no light which has traversed the crystal in the direction of the axes can emerge into the air. In this case the measurement can be performed in a fluid contained in a vessel with two plane glass faces perpendicular to the line of vision. The process is otherwise the same as before. Let the axial angle here observed be a', we then find a, if N is the index of refraction of the fluid, from the equation

$$sin\ \tfrac{1}{2}a = N\ sin\ \tfrac{1}{2}a'$$

Since the angle of the axes is different for different colours accurate measurement requires light of definite colour, *e.g.* that of the sodium flame or that produced by passing the light through glass coloured red by copper. The displacement of an axis when observed in different colours is called the *dispersion of the axes* for these colours.

The measurement of the same axial angle (*e.g.* in sulphate of baryta), in air a, and in a fluid a', affords a convenient means of determining the refractive index N of the fluid, with the help of the equation given above.

47A.—PHOTOMETRY.

I. *By Illumination at varying Distance.*

If two sources of light at the respective distances a_1 and a_2 produce the same illumination on a surface, the intensities of the lights are in the proportion $i_1 : i_2 = a_1^2 : a_2^2$. For estimating the equal illumination the following means are used.

(1.) *Shadow Photometer* (Rumford).—A dark rod not too

thin is placed upright before a white screen. The sources of light are so placed that the two shadows of the rod are quite close to each other. The distances are then so regulated that the two shadows appear equally dark, taking care that the rays of light from both sources reach the screen in the neighbourhood of the shadows at the same angle. The distances are of course measured from each light to the shadow cast by the other.

(2.) *Illumination of Two Surfaces.*—Two equal portions of a surface are illuminated under equal angles by the two sources of light, of which the distances a_1 and a_2 are so proportioned that the brightness of the surfaces appears equal (Foucault). Stray light must be shut out in this experiment. Either the surfaces are inclined to each other, illuminated from the outside and observed from the middle line (Ritchie), or they are separated by a partition and the transmitted light compared.

In Leonh. Weber's photometer two pieces of milk glass are illuminated, the one by a constant (Benzine) flame, the other by first one and then the other of the lights to be compared. A prism with total reflection brings the images of the glasses near each other. By regulating the distances equal brightness is produced. This photometer may also be used for the measurement of the illumination of surfaces in any position by making the distance of the benzine lamp variable (*Wied. Ann.* xx. 326, 1883).

(3.) *Comparison of Reflected and Transmitted Light* (Bunsen). —A small paper screen made unequally translucent in different parts, either by means of a circular, or, better still, ring-shaped spot of grease or stearin, or even by covering a part of a thin paper with a second thickness.

On one side of the screen at a fixed distance is a constant source of light such as a candle, small gas flame of constant height, or benzine or petroleum lamp which has been lighted a quarter of an hour previously. The two lights to be compared are successively placed on the other side of the screen at such distances, r_1 and r_2, that the different parts of the screen appear equally bright when looked at from the same direction.

A better arrangement than the paper with a spot of grease

is a combination of two prisms, which in a central circle where the glasses are plane and in contact allow the light to pass through whilst outside the circle total reflection occurs.

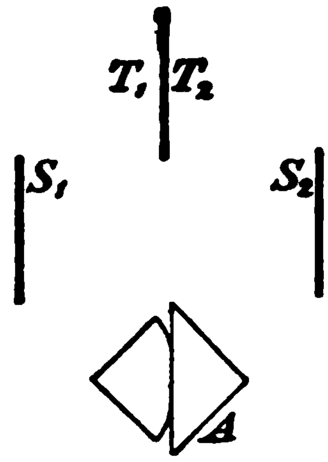


Fig. 35.

The lights to be compared are placed right and left of the screen T, which is equally white on the two sides. S_1 and S_2 are similar mirrors. The side T_1 of the screen is seen through the prisms on looking through the face A, the surface T_2 on the contrary, is seen by total reflection in the front prism. The circle becomes invisible when T_1 and T_2 are equally brightly illuminated (Lummer and Brodhun).

Cf. Zeitschr. für Instrum-kunde, ix. 44, 1889; xii. 41, 1892.

Comparison of Lights of very Different Intensity.—The two sources of light are each compared with a lamp of which the brightness is most suitably about the geometrical mean of those of the two lights. The two ratios are then multiplied together.

II. *By Polarisation.*

If the plane of vibration of polarised light passing through a polarising apparatus makes an angle ϕ with the plane of vibration of the latter the fraction $cos^2\phi$ will be transmitted if we neglect the loss by reflection (Malus).

(1.) Let the one half of a field of view be illuminated by a constant source of polarised light, the other half by a less bright source of light which is to be compared with the former. Let this field of view be observed through a Nicol's prism, and let the halves appear equally bright when the plane of polarisation of the prism makes an angle ϕ_1 with the vibration plane of the polarised beam. The other light is then substituted at the same distance. Let the angle ϕ_2 produce equality in the two halves. Then

$$i_1 : i_2 = cos^2\phi_1 : cos^2\phi_2$$

(2.) Two beams of light are polarised in planes at right angles to each other, and are arranged to illuminate the two halves of a field of view which is observed through a Nicol's

prism capable of rotation. This is turned until the two halves are equally bright. If ϕ_1 and $\phi_2 (= 90 - \phi_1)$ be the angles which the vibration plane of the Nicol makes with those of the two beams of light, we have $i : i_2 = cos^2\phi_2 : cos^2\phi_1 = tan^2\phi_1$. Errors may be recognised and eliminated by reversing the positions of the sources of light (Zöllner).

(3.) Equal quantities of light polarised in planes at right angles to each other behave when mixed together as ordinary light. The equality of the two parts can therefore be recognised through the polariscope (*e.g.* Savart) by the disappearance of the interference phenomena (Arago; Wild, *Pogg. Ann.* cxviii. 193, 1863).

Absolute Unit of Light.—An invariable source of light has not yet been completely attained. The old "normal candle" of spermaceti is still used, of which the flame should have a certain height (English candles 45 mm.) and the consumption of spermaceti during the experiment is controlled by the balance. Hefner's lamp burning amyl acetate is, however, now frequently used. With a wick 8 mm. thick it gives a flame 40 mm. in length. The intensity of the light is affected by the amount of carbonic acid in the air (*Electrotech. Z. S.* 1884, p. 20).

The light emitted by 1 sq. cm. of platinum just melting (Violle) is of course a normal unit but seldom available; it is much to be desired that a sufficiently simple and trustworthy form might be found for it.

III. *Intensity of Coloured Light in the Spectrum.*

According to Vierordt.—Instead of the scale in the spectroscope (**41**) there is an aperture movable laterally with a constant source of light (paraffin lamp) behind it, the white image of the aperture being reflected by the prism face upon the part of the spectrum to be examined. Of the light which falls on it such a fraction is allowed to pass through that the image just becomes invisible on the spectrum. The intensity of the light in any particular part of the spectrum is proportional to this fraction.

The subduing of the light is effected by a suitable combina-

tion of dark glasses, of which the transmitting power has been determined by I., 1 or 2. If some of the glasses have separately the transmissions d_1, d_2, d_3, etc., they have placed one behind the other $d_1 \times d_2 \times d_3 \times \ldots$

The above method is simple as to the means employed, but of course imperfect as to the results. If the slit of the spectroscope consists of two parts which can be separately varied in width and the respective widths measured, two spectra in contact may be obtained from two sources of light to be compared. By so regulating the widths of the two halves of the slit that at any place the brightness of the spectra is the same, the intensities of the lights for this colour of the spectrum are nearly proportional to the widths of the slit. Great differences of light are previously diminished by dark glasses.

Cf. Vierordt, *Pogg. Ann.* cxxxvii. p. 200, 1869; cxl. p. 172, 1870.

The more complete methods mentioned in II. 2 and 3, can also be used for the comparison of the intensities of particular colours in the sources of lights, by the introduction of prisms. This is carried out in the spectrophotometers of Glan (*Wied. Ann.* i. p. 351, 1877), and Wild (*ib.* xx. p. 452, 1883).

Glan's photometer has a slit divided into an upper and lower part. Through these two parts enter the two beams of light which are to be compared, one of them thrown on to its part of the slit by a total reflecting prism. In their passage through the tube of the collimator, the two beams are polarised in planes at right angles to each other, the other components being screened off.

In the telescope the spectra of the two lights appear one above the other; by a proper drawing out of the collimator tube they are brought into contact at any desired colour, and by movable screens the other parts of the spectra are hidden. A scale permits the description of the colours by numbers, just as in the spectroscope.

In front of the collimator tube is placed a graduated circle with a Nicol capable of rotation. This is adjusted so that the two halves appear equally bright. If this takes place with a rotation of ϕ, from the zero of the Nicol, the ratio of intensity for the two lights is $= k\ tan^2\phi$, where k is a factor

not differing much from 1, which results from the unequal diminution of the two beams in the instrument.

Otherwise the ratio would be

$$\cos^2(90-\phi)/\cos^2\phi=\tan^2\phi$$

The zero position of the Nicol can be found by the maximum darkness produced in one of the halves of the field of view. The factor k is found by producing equal brightness of the two halves when illuminated by the same flame behind very homogeneous milk glass. The angle of rotation required in this case being ϕ_0, $k=\cot an^2\phi_0$.

IV. *Determination of a Coefficient of Absorption with the Spectrophotometer.*

If out of a quantity of light s, passing through a body of the length δ, the small quantity σ is absorbed, $\frac{1}{\delta}\cdot\frac{\sigma}{s}=A$ is called the coefficient of absorption of the body for the light used. A depends upon the colour. By passage through a thickness d, the intensity i of the incident light is reduced to

$$i'=i\,.\,e^{-A\,.\,d}$$

If we measure i/i' we therefore find

$$A=\frac{1}{d}\,\log(nat)\frac{i}{i'}=\frac{1}{d}\,.\,2{\cdot}30\,\log\frac{i}{i'}$$

using the ordinary logarithms in the second formula.

i/i' is measured as above, by passing the original light through the one part of the slit, while the other part is covered by the absorbing body.

Loss of Light in Reflection.—With every passage through the surface dividing two media of different refractive power, there is a loss of light, since of the light incident perpendicularly, the fraction $\left(\frac{\mu-1}{\mu+1}\right)^2$ is reflected; where μ is the refractive index between the media. For glass therefore

$$[(\mu-1)/(\mu+1)]^2=(0{\cdot}5/2{\cdot}5)^2=\frac{1}{25}$$

When μ is known the losses can be calculated and the corrections arising can be introduced. They can also be avoided by making them practically equal for the two paths of the light. For this purpose the two halves of the slit are covered with layers of the body of unequal thickness, but similarly bounded, and the difference of thickness put for d in the calculation. In the case of absorbing glasses, one-half may also be covered with a thin colourless glass, and the loss then neglected.

47B.—Production of Elliptically Polarised Light, and Investigation of State of Polarisation (Babinet's Compensator).

Form of Vibration of the Light after passing through a Plate of Crystal.

Plane-polarised light, *i.e.* vibrating in only one plane, of wave-length λ, traverses a plate of a double refracting crystal not in the direction of an optic axis. The vibration is divided in the crystal into two components vibrating at right angles to each other. The planes passing through the direction of the ray, and also containing these components, will be called the principal planes.

(1.) If the plane of vibration of the incident light coincides with one of the principal planes, the light is not altered.

(2.) It is also unaltered at emergence if the difference of phase of the two vibrations produced in the crystal amount to λ, 2λ, 3λ, etc.

(3.) Let the difference of phase amount to $\frac{1}{2}\lambda$, $\frac{3}{2}\lambda$, etc.; in this case the emergent light is plane-polarised, but in general in a plane different from that of the incident light. If the plane of vibration at incidence make the angle ϕ with a principal plane, the plane of vibration of the emergent light will make the same angle ϕ with that principal plane, but on the opposite side. Special case: $\phi = 45°$; the emergent light has its plane of polarisation turned through 90°.

(4.) Let the plane of vibration of the incident light be inclined 45° to a principal plane. Let the difference of phase be $\frac{1}{4}\lambda$, $\frac{3}{4}\lambda$, $\frac{5}{4}\lambda$, etc. The emergent light is circularly polarised, *i.e.* the paths of the points of the ether are circles. $\frac{1}{4}\lambda$, $\frac{5}{4}\lambda$, etc., produce a circular movement in a direction contrary to that produced by $\frac{3}{4}\lambda$, $\frac{7}{4}\lambda$, etc. Similarly the direction of vibration is reversed when the plane of vibration of the incident light is altered by 90°.

(5.) In all other cases elliptically polarised light emerges. If the inclination of the plane of vibration to the principal planes amount to 45° (*cf.* 4), the ellipses are the narrower the nearer the difference of phase is to 0, $\frac{1}{2}\lambda$, λ, $\frac{3}{2}\lambda$, etc., and the more nearly circular the nearer the difference is to $\frac{1}{4}\lambda$, $\frac{3}{4}\lambda$, $\frac{5}{4}\lambda$, etc. See also fig., p. 211. The ratio of the axes of the vibration ellipse for the difference of phase $k\lambda$ is

$$\frac{a}{b} = \tan k\pi$$

I. *Production of Elliptically Polarised Light with the Compensator.*

Babinet's compensator gives the power of interposing in the path of a ray of light a plate of crystal of any desired effective thickness, and hence conferring upon the light different vibration forms.

For this purpose two very acute quartz wedges of equal angle, and equally thick in the middle, are placed one behind the other. The sharp edges are laid parallel to each other on opposite sides. The two wedges have the optic axis parallel to one face, but in one case parallel to the edge, in the other at right angles to it. The longer wedge is movable by a micrometer screw in the direction of its length, the movement being read off on a divided head. On looking through an eyepiece Nicol with its plane of polarisation (longer diagonal of the rhombus) inclined at an angle of 45° to the principal section of the quartz, and always (by means of a wire in the eyepiece) in the direction of the middle of the fixed quartz wedge, the light emerges unaltered when equal thicknesses of the two wedges are at this place superimposed. By moving the one wedge a difference of phase in the two components proportional to the displacement is produced.

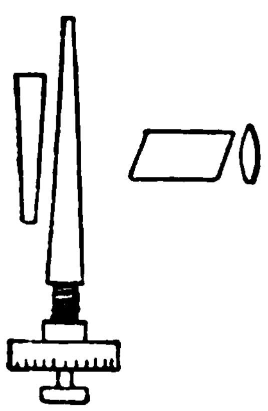


Fig. 36.

To determine first the value ϵ of a division of the graduated head (of course for light of some definite wave-length) expressed as difference of phase, the incident light is passed through a Nicol with its polarisation plane also at an angle of 45° with the principal section, and the observation made through the

eyepiece Nicol, the prisms being "crossed." Somewhere in the field of view a dark band will then appear, which by means of moving the wedge is brought upon the wire. Let now the reading of the divided head be p_0. Now the head is turned until the next dark band is upon the wire. Let the reading now, of course counting the whole revolutions, be p_1. Since between p_0 and p_1 the difference of phase has obviously been altered by one whole wave-length λ, the displacement by one division means a difference of phase of $\lambda/(p_1 - p_0)$ and

$$\epsilon = \frac{1}{p_1 - p_0}.$$

The dispersion of the quartz being small, ϵ is nearly proportional to the wave-length of the light employed.

The figure annexed depicts the vibration state of the light in the range between two dark bands for the case that the incident light vibrates from the left above to right below (*i.e.* according to Fresnel, that the short diagonal of the polarising Nicol is so situated). In the dark band the light is unaltered (2); midway between the bands, on account of the acceleration by $\frac{1}{2}\lambda$, plane-polarised light with its plane turned through 90° is produced (3).

At $\frac{1}{4}$ distance from each an acceleration of $\frac{1}{4}\lambda$ and $\frac{3}{4}\lambda$ is produced, and therefore circularly polarised light.

In the one half the vibration is left-handed, in the other right-handed.

The figure corresponds to the case in which, passing from left to

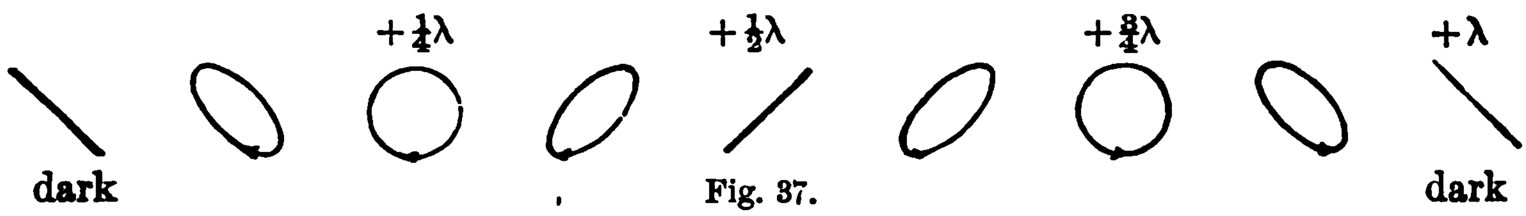


Fig. 37.

right, the horizontal component of the vibration is retarded with respect to the vertical. But since in quartz, as a positive crystal, the extraordinary ray, viz. that vibrating parallel to the optic axis, is the most retarded, the edge of that wedge of which the optic axis lies in the length of the wedge is to the left.

Ratio of the Axes of the Ellipse.—The ellipticity of a beam of light is determined by the ratio a/b of the two principal axes of the ellipse. When the divided head reads p there is at the cross-wires light of the axis ratio (see above, 5)—

$$\frac{a}{b} = \tan\left[\epsilon \cdot 180^\circ(p - p_0)\right]$$

Determination of the Zero-Point.—At the dark places the difference of phase is a whole multiple of the wave-length; but whether 0, λ, 2λ, etc., cannot be determined in homogeneous light. In order to find that band in which the difference of phase is equal to 0, *i.e.* where the quartz wedges are equally thick, we need only use white light. Then only one really dark band is found which determines the zero-point. The others are coloured on account of the differing wave-lengths of the various colours.

II. *Investigation of the Vibration Form of a Beam of Light.*

We assume the light to be homogeneous and of the wave-length taken above, for which ϵ was determined. Further, this light is not, for instance, ordinary light mixed with polarised, but light with a definite vibration ellipse as produced from plane polarised light by passage through some crystal such as mica.

Position of the Axes of the Vibration Ellipse.—Let the compensator be adjusted with the divided head accurately at the point $\frac{1}{4}\lambda$ (fig., previous page), at which plane polarised light is converted into circular polarised—in other words, at the division $p_0 + \frac{1}{4}(p_1 - p_0)$. Let the compensator be now directed to the light to be investigated. Let the compensator be capable of rotation round its visual axis. On thus rotating the dark band generally moves; let the rotation be continued till it falls on the wire. The two principal sections of the comparator coincide in this position with the two axes of the vibration ellipse. The analysing Nicol is rotated so that the bands are maintained as dark as possible. If at no position of the comparator does a band reach the wire the analyser is turned through 90° when such will be the case.

Special Cases.—(1.) If the field of view does not alter when the whole compensator is rotated, circular polarised light is indicated. If in this case the analyser is rotated alone, a dark band coinciding with the wire alternates with two occurring at equal distances on each side. (2.) On the other hand, if on rotating the whole compensator, an alternate occurrence of bands symmetrically right and left of the wire are observed instead of moving bands, this shows plane polarised light.

Ratio of the Axes of the Ellipse.—In order to determine this we carefully rotate the compensator into that position at which the band is as dark as possible. Let the plane of polarisation in the analyser form an angle with the principal section of the compensator of ϕ, then

$$\frac{\text{Axis parallel to this principal section}}{\text{Axis perpendicular to this principal section}} = \tan \phi.$$

(On the theory of Babinet's Compensator, see *e.g.* Dorn, in Appendix to Neumann, *Vorlesungen über Optik;* C. Schmidt, *Zeitschr. f. Instr.-Kunde*, 1891, p. 439.)

METHODS OF OBSERVATION FREQUENTLY EMPLOYED IN MAGNETIC AND ELECTRICAL WORK.

48.—Angular Measurement with Telescope, Mirror, and Scale.

This method may be employed with great advantage in many magnetic and galvanic observations, but its application is limited to the measurement of small angles.

A small vertical mirror is attached to the suspended magnet, etc., of which the horizontal deflection is to be measured, and, in order to simplify calculation, it should be near the axis of rotation of the latter. At a distance of from 1 to 5 meters from the magnet is fixed a horizontal scale at the same level as the mirror; and its reflected image is observed with a telescope provided with cross-wires. The scale must be so placed that when the magnet is in its position of equilibrium, to which the other positions are mostly referred, that point of the scale from which a perpendicular would cut the mirror shall be visible on the cross-wires of the telescope. We may call this point briefly the "middle scale division," and the corresponding position of the magnet its "mean position."

Arrangement of the Telescope and Scale.—The telescope is first focussed approximately for twice the distance between the mirror and scale. It is then pointed to the mirror, and so placed that its objective is visible, in the mirror, to an eye immediately over the middle scale division, or conversely that this is seen from near the telescope. The image of the scale will then be visible in the telescope, or will appear by a slight movement of the latter. Lastly, the fine adjustments

must be made; the cross-wires must be clearly focussed, and the telescope then drawn out till the scale and cross show no parallax; that is, till their relative position is unaltered by moving the eye before the eyepiece.

If observers requiring different foci take turns at reading, clear definition must be obtained in each case by moving only the eyepiece between the eye and the cross-wires. In the selection of a telescope care must be taken that the eyepiece is adjustable in this way by screwing or sliding.

The measurement of angles with mirror and scale may also be carried out by the use of a well-defined source of light (slit, thread in front of a flame), which is reflected from the mirror on to a scale, a lens being placed so that the rays pass through it both in passing to and from the mirror. [The focal length of the lens should be equal to the distance between the scale and the mirror.] When the lens is correctly adjusted, a clear image of the mark is obtained on the scale, the displacement of which is made use of in the same way as the image in the telescope. [This plan has also the advantage of being easily seen from anywhere in front of the scale, and therefore by many people at the same time.]

Receipt for Silvering Glass (after Boettger).

(1.) 5 grm. argentic nitrate is dissolved in distilled water, and ammonia added to the solution till the precipitate first thrown down is almost entirely redissolved. The solution is filtered and diluted to 500 c.c.

(2.) 1 grm. of argentic nitrate is dissolved in a little water, and poured into half a liter of boiling-water; 0·83 grm. of Rochelle salt is added, and the mixture is boiled for a short time, till the precipitate contained in it becomes gray, and is then filtered hot. It may be kept in the dark for some months.

The glass plates, thoroughly cleaned (with nitric acid, caustic soda, alcohol), are placed in a shallow vessel, and covered a few millimeters deep with equal volumes of the two solutions, or still better, suspended in it face downwards. In an hour the reduction will be complete; the plates are rinsed and the operation repeated until a sufficient coating of silver is obtained. When the silvered surfaces are dry, they may be cautiously polished with the palm of the hand. If the silver be only required as a coating of the back surface, the polishing is of course superfluous. In this case also the operation may be shortened by heating the second solution to about 70° C. before mixing. The silver may then be varnished

over as a protection, but thin mirrors are apt to be warped by the contraction of the varnish.

The thin glasses used for covering microscopic objects make good mirrors, but those only which give a clear image of the scale can be employed.

Other recipes with sugar-candy and caustic alkaline solutions (free from carbonic acid!) are given by Martin, *Pogg. Ann.* cxx. 335, 1863; and Lohse, *Jährbuch für Photographie,* 1887.

49.—Reduction of Observations with the Scale to Angular Measure.

We will reckon all angles of rotation from the "mean position" (see p. 214) as zero, and denote by ϕ the angle of deflection through which the magnet, etc., is turned from this position. As scale deflection, we take the difference n of the observed from the middle scale division.

(1.) For small deflections the angle is proportional to the scale-reading; and, indeed, if A be the distance of the reflecting surface from the scale, expressed in scale divisions, (millimeters, if it be a millimeter scale), the value of 1 division in degrees of arc is

$$= \frac{28^\circ{\cdot}648}{A} = \frac{1718'{\cdot}9}{A} = \frac{103132''}{A}$$

Also

$$sin\ \phi = tan\ \phi = \frac{e}{2A}.$$

(2.) For greater deviations we may employ the series

$$\phi = \frac{28^\circ{\cdot}648}{A} e \left(1 - \tfrac{1}{3}\frac{e^2}{A^2} + \tfrac{1}{5}\frac{e^4}{A^4}..\right)$$

$$tan\ \phi = \frac{e}{2A}\left(1 - \tfrac{1}{4}\frac{e^2}{A^2} + \tfrac{1}{8}\frac{e^4}{A^4}..\right)$$

$$sin\ \phi = \frac{e}{2A}\left(1 - \tfrac{3}{8}\frac{e^2}{A^2} + \tfrac{31}{128}\frac{e^4}{A^4}..\right)$$

$$sin\ \frac{\phi}{2} = \frac{e}{4A}\left(1 - \tfrac{11}{32}\frac{e^2}{A^2} + \tfrac{431}{2048}\frac{e^4}{A^4}..\right)$$

For deflections not exceeding 6°, the first term of the correction is usually sufficient.

Hence we reduce a scale-reading e to the corresponding

arc, tangent, sine, and sine of half-angle, by subtracting $\frac{1}{3}$, $\frac{1}{4}$, $\frac{3}{8}$, or $\frac{11}{33}\,\frac{e^3}{A^2}$ respectively from e.

(3.) For deflections of any magnitude whatever

$$tan\ 2\phi = \frac{e}{A},\ \text{or}\ \phi = \tfrac{1}{2}\ tan^{-1}\frac{e}{A}$$

The last formula is given by simple geometrical considerations, the others by taking the first two terms only of the series for the development of ϕ, *tan* ϕ, etc.

In Table 21A corrections for reduction to arc are given for different distances between mirror and scale. In order to reduce to tangents, the amount of the correction must be reduced one-fourth. The table is most conveniently used in a graphic form, so that the value for any given deflection can be deduced from the curve.

The measurement of distance between scale and mirror with an accuracy of 1 mm. presents no difficulty either with a steel tape, a wire which is afterwards compared with a scale, or with two rules which are slid one over the other. For more exact measurement, two short contact rules may be employed, of which one touches the mirror and the other the scale, and from which, by means of fine wire, we plumb down on to a sufficiently long rule, or to two points, on the floor of which the distance may be exactly measured.

Paper-scales are liable to alter in length with time to a material extent, and milk-glass mm. scales, such as those of Hartmann and Braun, are much to be preferred.

Corrections required under various Circumstances.

(*a.*) *For the Thickness of Covering Glass.*—If a glass plate of thickness d and refractive index n be placed in the path of the ray, it is necessary to deduct $d(n-1)/n$ or for ordinary glass about $\frac{1}{3}d$. (Compare **39**A, 1.)

(*b.*) *For Thickness of Mirror.*—The distance from the scale of a glass mirror, silvered on the back, is not that measured from the front, with the actual thickness of the glass added, but only its optical thickness $\frac{\delta}{n}$, or approximately $\frac{2}{3}\delta$. If it

is impossible to measure the thickness with a rule, this "optical thickness" may be measured direct as half the distance of a point on the surface from its image in the mirror. (Compare **39A**, 3.)

(*c.*) *For Inclination of the Mirror.*—Calling the height at which the vertical plane of the scale is intersected by the normal of the mirror N, that at which it is intersected by a horizontal passing through the mirror H, and that of the line of sight of the telescope F, the actual scale-distance A_0 must be calculated from the measured horizontal distance of the scale from the mirror, as

$$A = A_0 + \frac{(N-H)(N-F)}{A_0}$$

(*d.*) *For Curvature of the Mirror.*—If a curved mirror be attached at a distance of a from the axis of rotation, the measured distance A must be increased by $A_0 a/r$ if r be the radius of curvature of the mirror (**43**, III.). For convex mirrors r is negative. Since mirrors are easily deformed by the pressure of their fastenings, this correction may become important in cases of considerable excentricity from the axis of rotation.

50.—Determination of the Position of Equilibrium of a Swinging Magnetic Needle.

The *position of equilibrium*, or point of the scale on which a magnetic needle would settle when it came to rest, may be determined from observations of the moving needle in the following manner:—

(1.) *Observation of Turning-Point.*—If the oscillations are rapid or large, a number of successive turning-points of the cross-wires (*i.e.* points at which the direction of motion is reversed) are observed on the scale. From any three of these the position of equilibrium may be found by taking the arithmetical mean of the first and third, and again that of the second and the number so obtained. Compare, besides, the article (**7**) on the determination of the position of equilibrium of a balance, which is completely applicable to the present case.

(2.) *Observation of Position.*—If the motion of the needle

be so slow that the position of the cross-wires on the scale can be exactly observed at any moment, the arithmetical mean between two successive readings, differing by the time of one oscillation, will give the position of equilibrium.

(3.) *Damped Magnetic Needles.*—These two rules are only applicable when the amplitude of swing diminishes very slowly. If, however, the magnet be damped and rapidly brought to rest (*e.g.* by surrounding it with a copper frame), the position of equilibrium p_0 is found from two successive observations, p_1 and p_2, differing by the time of one oscillation, by the following formula:—

$$p_0 = p_2 + \frac{p_1 - p_2}{1 + k}$$

Here k is the "ratio of damping," that is, the ratio of one arc of oscillation to the next following. Compare the example in the following article. The reduction of scale-readings to angular measure is rarely necessary.

To bring the needle to rest, a magnet is frequently employed, which is approached or withdrawn at the same level as the needle. A galvanic current passing near the needle, and closed and broken at the right moments, may be employed for the same purpose.

51.—Damping and Logarithmic Decrement of a Magnetic Needle.

The diminution of the arcs of oscillation of a magnetic needle which is damped by a copper case, or by the surrounding coils of a multiplier, is of great importance in galvanic and magnetic measurements. The damping is caused by the reaction of the currents induced in the neighbouring conductors by the moving needle; and the law of damping, given by the theory of induction, shows that the arcs diminish in a geometrical series. The constant relation k of an arc of oscillation to that next following is called the ratio of damping, and the logarithm λ of the latter the logarithmic decrement of the needle (Gauss).

The determination of this magnitude is most simply accom-

plished by observation of a series of turning-points of the needle. The difference of two successive turning-points, which, if the oscillations be large, must be corrected to angular measure (article **49**), gives the arc. If a_m be the amplitude of the mth, and a_n that of the nth oscillation,

$$k = \left(\frac{a_m}{a_n}\right)^{\frac{1}{n-m}} \text{ and } \lambda = \frac{log.\,a_m - log.\,a_n}{n-m}$$

Errors of observation have the least influence on the result when $\frac{a_m}{a_n}$ is about 2·8.

From a longer series of observations (best an uneven number) the required magnitude may be deduced as in the following example, in which 7 observations of turning-point are contained in the first column. The second column gives the distance of the turning-point from the middle scale division (in this case 500); the third and fourth the correction (article **49**), to reduce the scale-readings to numbers proportional to their angular values. The distance of the scale from the mirror is 2600 scale divisions; and the correction to reduce the deflections to arc is therefore $\frac{1}{3}e^3/2600^2$ (**49**). In column 5 are the six corrected arcs, combinations of the first and fourth, second and fifth, etc., of which each gives a value for k and λ. In the sixth and seventh columns is shown the method (**50**, 3) of calculating the position of equilibrium from the two turning-points when the ratio of damping, $k = 1{\cdot}151$, is known.

Example—

Observed turning-points.	n	$\frac{n^3}{3 \times 2600^2}$	Corrected turning-points.	Arcs a	$\frac{a}{2{\cdot}151}$	Position of Equilibrium.
285·0	215	0·5	285·5			
710·0	210	0·5	709·5	424·0	197·1	512·4
341·2	159	0·2	341·4	368·1	171·1	512·5
662·5	162	0·2	662·3	320·9	149·2	513·1
383·9	116	0·1	384·0	278·3	129·4	513·4
625·7	126	0·1	625·6	241·6	112·3	513·3
415·6	84	0·0	415·6	210·0	97·6	513·2
						Mean = 513·09

We obtain therefore—

from 1	and 4,	$\lambda = \frac{1}{3}$	(*log.* 424·0	− *log.* 278·3)	= 0·0610
„ 2	„ 5,	„	368·1	241·6	0·0609
„ 3	„ 6,	„	320·9	210·0	0·0614

$$\text{Mean } \lambda = 0{\cdot}0611$$
$$k = 1{\cdot}151$$

With increasing angle of oscillation, the damping somewhat diminishes, and in a ratio approximately proportional to the square of the angle. This is the more noticeable, the narrower and higher the damper or multiplier, and the longer the swinging magnet. A part of the damping is always dependent on the resistance of the air. If the damping be required which is due to the multiplier alone, one series of observations must be made with open, and another with closed circuit. The logarithmic decrement of the former subtracted from that of the latter gives the required decrement due to the multiplier alone.

By employing natural logarithms or multiplying the value of λ as found above by 2·3026, we obtain the "natural logarithmic decrement." (Compare also **78.**)

52.—Period of Oscillation.

The time of oscillation of a body oscillating about its position of equilibrium is the time between one elongation (turning back, greatest deviation from the point of rest) and the next on the opposite side.* The instant of turning, however, is unsuitable for direct observation, as at that moment the motion of the body is insensible. On the other hand, it passes a point near its position of equilibrium with the greatest velocity, so that the instant of this crossing may be exactly observed. From the times of two successive passings of the same point in opposite directions, that of the intermediate turning-point is simply found as the arithmetical mean.

A point near the position of equilibrium is marked on the scale (by hanging over it a dark thread), and the times at which it is passed are observed by the ticking of a

* In Acoustics and Optics double this interval is sometimes taken as the period of oscillation.

seconds clock. The mean of each successive pair of observations is taken, and the differences between these means give the time of oscillation. The tenths of seconds are estimated by the relative distances of the cross-wires from the mark at the ticks of the clock preceding and following the passing.

If from a consecutive series of n values so obtained the mean be again taken, it will only yield the same result as if the difference of time between the first and last observation were divided by n. The intermediate observations will therefore be useless. To render the whole available, the observations may be divided into two parts, and the differences of the corresponding numbers in the two halves taken, and from these the arithmetical mean reckoned and divided by $\frac{1}{2}n$.

Example—

Time of crossing (observed). m.	sec.	No.	Time of turning (calculated). m.	sec.	Time of oscillation.	sec.
10	3·3					
	16·5	1	10	9·90	from Nos. 1 and 4, $\frac{39{\cdot}90}{3}$	$=13{\cdot}30$
	29·9	2		23·20		
	43·0	3		36·45		
	56·6	4		49·80	„ 2 „ 5, $\frac{40{\cdot}05}{3}$	$=13{\cdot}35$
11	9·9	5	11	3·25		
	23·3	6		16·60	„ 3 „ 6, $\frac{40{\cdot}15}{3}$	$=13{\cdot}38$
					Mean	13·34

(For the use of the method of least squares in such observations, see **3**, II.)

It is best of all to obtain two widely-separated and exactly-determined times of elongation from repeated observations, in the following manner:—We observe twice (or for great accuracy even more frequently) an even number of successive times of passing the marked point, and from each pair lying symmetrically about the middle elongation we take the arithmetical mean, and from these, again, the mean of the whole.

Example—

	First Set.			Second Set.	
No.	Times of passing.	Nos.	Means.	Times of passing.	Means.
	m. sec.		m. sec.	m. sec.	m. sec.
1.	7 40·7			10 10·5	
2.	49·0			18·9	
3.	55·6	3. 4.	7 59·80	25·6	10 29·75
4.	8 4·0	2. 5.	59·85	33·9	29·75
5.	10·7	1. 6.	59·75	40·6	29·70
6.	18·8			48·9	
	Mean of whole		7 59·80		10 29·73

These two means are the times of two elongations, as exactly as they can be deduced from these observations. The difference between them, 149·93 sec., divided by the number of intermediate oscillations, gives the time of oscillation. It is not necessary actually to count these oscillations, as the number may be deduced from the observations themselves. An approximation to the time of oscillation is easily obtained from either series. Taking, for example, the first: from the first and last pairs of observations are obtained the times of two elongations—viz. 7 m. 44·8 sec. and 8 m. 14·7 sec., between which four oscillations have occurred. Hence the time of oscillation is $\frac{29{\cdot}9}{4} = 7{\cdot}47$ sec. If this number and the observations were perfectly exact, 7·47 would divide 149·93 without a remainder, and the quotient would be the number of oscillations sought. Performing the division we find 20·07, a value so near to the whole number 20 as to leave no doubt that this is the number of oscillations in 149·93 sec. The exact time of oscillation is therefore

$$\frac{149{\cdot}93}{20} = 7{\cdot}496$$

In the estimation of the number of oscillations between the sets, the care required will naturally increase with the number, and, other things being equal, with the rapidity of the oscillation. The possibility of an error will be diminished if we observe at each passage whether the motion corresponds to a

greater or lesser period, and also by our accustoming ourselves always to begin with a passage in the same direction. The required number of oscillations will then necessarily be even.

In order to eliminate errors of observation, a large even number, $2m$, of sets of observations may be made, and No. 1 combined with $m+1$, 2 with $m+2$ .... m with $2m$, and the mean of the single results taken.

This method obviously requires that the oscillations should be sufficiently slow for the time of each to be observed (many observers will find this difficult even with a period of 7 sec.). It may, however, be employed for more rapid oscillations, by each time omitting 2 (or any even number of) passings, and forming the set, for instance, of Nos. 1, 4, 7, 10, 13, and 16, which are reckoned precisely as above, except that the result is of course divided by 3.

Short oscillations, of a period not exceeding a few seconds, are more conveniently observed at the turning-point, than in passing the centre, and best at one side only, and if necessary, omitting intermediate oscillations, as described in the preceding paragraph.

The time of oscillation of a "damped" needle with the logarithmic decrement λ is to that without damping as $\sqrt{\pi^2+(2{\cdot}306\lambda)^2}$ to π (**51**, **78**).

It is manifestly unimportant to the method whether the observations are made with mirror and scale, or with the naked eye.

If the time of oscillation be very near a second, or an exact multiple or sub-multiple of one, the method of coincidences may be employed. In this case, the times must be noted at which the passage of the position of equilibrium exactly coincides with the beat of a seconds clock. The time of oscillation is then given by dividing the number n of seconds between two such coincidences by $n+1$ or $n-1$, according to whether the oscillations are quicker or slower than those of the pendulum (compare **3**, II. on the calculation).

If a watch or chronometer be employed instead of a clock, it is convenient to count 5 or 10 ticks after the passage of the marked division before noting the time, so as to allow time to look from the telescope to the watch. If an absolute time be wanted, this

must, of course, be subtracted from the mean result. A spot of light reflected on the scale from a lamp (as in Thomson's galvanometers) is often conveniently substituted for the telescope.

Reduction of the Time of Oscillation to that in an infinitely Small Arc.

In the form of oscillation common to magnets, bodies with bifilar suspension, the ordinary pendulum, and in general to all cases where the moment of rotation is proportional to the sine of the angles of deflection, the time of oscillation increases slightly with the amplitude. As we usually require the limiting value to which the time approaches when the oscillations are very small, we must apply a correction to the observed values which are obtained from larger amplitudes.

Taking

t = the observed period of oscillation;
α = the arc through which the magnet vibrates;

the time of oscillation, in an infinitely small arc, is—

$$t_0 = t - \left(\tfrac{1}{4}\sin^2\frac{\alpha}{4} + \frac{5}{64}\sin^4\frac{\alpha}{4}\right)t$$

To facilitate the calculation, the quantity within the brackets may be found in Table 21, calculated for arcs up to 40°; an amplitude which should never be exceeded.

The method of observation with the telescope and scale possesses the advantage that the oscillations (of from 50 to 300 divisions of the scale) are so small that the first term of the formula of correction is sufficient. We may therefore write, if

p = the arc of oscillation in divisions of scale;
A = the distance of mirror from scale; also expressed in divisions of scale—

$$t_0 = t - \frac{t}{256}\frac{p^2}{A^2}$$

The value of α (or p, as it is written in the above formula) may be taken as the mean of the arcs of the first and last observed oscillations. The observations must be so arranged that the amplitude does not diminish by more than one-third during the experiment.

If we call the mean of the first and last arcs of oscillation a, and their difference d, we may substitute for a or p with greater and always with sufficient accuracy

$$a\left(1 - \frac{1}{24}\frac{d^2}{a^2}\right)$$

The complete formula for reduction of time of oscillation to that in an infinitely small arc is—

$$t = t_0\left(1 + \tfrac{1}{4}\sin^2\frac{a}{4} + \frac{9}{64}\sin^4\frac{a}{4} \ \ldots\ldots\right)$$

The formula given above is obtained from this by performing the division, omitting all powers beyond the 4th, which is practically always admissible. The reduction formula for scale observations may readily be found with the help of article **49.**

53.—Bifilar Suspension (Harris, Gauss).

In order that a heavy body suspended by two threads may be in a position of equilibrium, the threads must both be in the same vertical plane. Let

e_1 and e_2 be the horizontal distances of the upper and lower ends of the two threads,

h the mean length of the threads. If the threads deviate from the perpendicular, h is the mean perpendicular distance of the upper and lower ends.

For small rotations of the bifilar body, the backward moment of rotation is proportional to the sine of the angle of rotation. If the length of the threads is very great in proportion to their separation, this is also true for larger deflections.

Let P be the sum of the vertical tensions of the threads. Then for the angle of deflection a, the backward moment of rotation is

$$P\frac{e_1 e_2}{4h}\sin a$$

P is the weight of the suspended body, plus half the weight of the threads. In the "Absolute" system of measure the

weight must be considered as mass, and multiplied by the acceleration of gravitation (App. 6 and Table 8A).

The tension of the two threads is equal when the centre of gravity of the suspended body lies in the mean vertical between them. This condition is tested by supporting the body on a point in the mean vertical, when its position ought not to be altered.

Stiffness of the threads has the same influence as shortening them. Let ρ be the radius, and E the modulus of elasticity. We must then deduct from the measured length

$$\delta=\rho^2\sqrt{\frac{2\pi E}{P}}$$

Since the ordinary modulus of elasticity is expressed in kilograms and square millimeters, that expressed in grams and centimeters will be 100,000 times greater, as, for instance, for iron $E=200\times10^7$, brass 90×10^7, etc., ρ in cm. and P in grm. gives δ in cm.

Elasticity of Torsion.—The moment of torsion of the two threads together amounts to (**36**)

$$\frac{2\pi\rho^4E}{5h}\cdot a$$

or, where a is small, to *sin* a. In the "Absolute" system this must be multiplied by the factor $g(=981\ cm./sec.^2)$. The total "directive force" to be multiplied by *sin* a is therefore

$$D=gm\frac{e_1e_2}{4(h-\delta)}+\frac{2\pi}{5}gE\frac{\rho^4}{h}$$

where m is the mass of the suspended body increased by half the mass of the threads.

Example.—The brass suspending wires of 300 cm. length, are 0·01 cm. thick, or $\rho-0{\cdot}005$. The bifilar body weighs 100 grm. Then

$$\delta=0{\cdot}005^2\sqrt{\frac{2\pi\times90\times10^7}{100}}=0{\cdot}19\ cm.$$

Again

$$\frac{2\pi}{5}gE\frac{\rho^4}{h}=\frac{2\times3{\cdot}14}{5}981\times90\times10^7\,.\,\frac{0{\cdot}005^4}{300}=2{\cdot}3\ (cm.^2g.sec.^{-2})$$

The wires together weigh 0·42 grm., therefore $m = 100 + 0{\cdot}21 = 100{\cdot}21$ grm. Lastly, let $e_1 = e_2 = 12$ cm.; then

$$gm\frac{e_1 e_2}{4(h-\delta)} = 981{\cdot}0 \times 100{\cdot}21 \frac{12 \times 12}{4 \times 299{\cdot}81} = 11829 (cm.^2 g.sec.^{-2}).$$

The total directive force amounts therefore to 11831 ($cm.^2 g.sec.^{-2}$). (Compare F. Kohlrausch, *Wied. Ann.* xvii. 737, 1882).

Directive Force from Observations of Oscillation.—If K, the moment of inertia of the bifilar body in relation to its axis of rotation, be known, the directive force D may be deduced from the period of oscillation t as (**54**, App. 10)

$$D = \pi^2 \frac{K}{t^2}$$

This method is advantageous in cases where the suspending wires are thick, or their separation small.

54.—Determination of Moment of Inertia.

The moment of inertia of a material point, referred to an axis round which it revolves, is $l^2 m$, where m = the mass of the point, and l its distance from the axis. That of a number of points rigidly connected, or of a body, is the sum or integral of those of all the individual points. It must of course be expressed by some units of length and mass. This is most briefly expressed by writing after the number for the moment of inertia [g.cm.^{2}] or [mg.mm.^{2}] (see Appendix 10).

I. *Calculation of Moment of Inertia.*

In bodies of regular form and homogeneous composition the moment of inertia may be found by calculation.

In the following formulæ, which embrace the more frequent cases, m is always the mass of the body, and K its required moment of inertia.

Thin Bar of length l, and of uniform width. Referred to an axis at right angles to the rod, and passing through its centre $K = m \cdot l^2/12$.

Right-angled Parallelopipedon.—a and b are two adjacent

edges. The moment of inertia round an axis passing through the centre of gravity, and parallel to the third edge (that is, perpendicular to a and b), is

$$K = m\,\frac{a^2 + b^2}{12}$$

Cylinder or disc of radius r referred to the axis of the cylinder—

$$K = m\,\frac{r^2}{2}$$

Referred to an axis perpendicular to the middle of the axis of the cylinder (l being the length of the cylinder)—

$$K = m\left(\frac{l^2}{12} + \frac{r^2}{4}\right)$$

Hollow cylinder of radii r_0 and r_1. Moment of inertia referred to the axis—

$$K = m\,\frac{r_0^2 + r_1^2}{2}$$

Referred to a line perpendicular to the axis—

$$K = m\left(\frac{l^2}{12} + \frac{r_0^2 + r_1^2}{4}\right)$$

Sphere of radius r, referred to a diameter—

$$K = m\,\frac{2}{5}\,r^2$$

Note.—If, as in the foregoing examples, the moment of inertia K, relative to an axis passing through the centre of gravity, be given, the moment of inertia K_1, relative to any other axis parallel to the first, may be obtained by adding to K the product of the mass of the body m and the square of the distance a between the new axis and the centre of gravity; that is—

$$K_1 = K + a^2 m.$$

II. *Determination by Loading* (Gauss).—The moment of inertia may be found experimentally in the following manner:—The time of oscillation t is observed, and the moment of inertia then increased by a known amount K', without altering the directive force, and the time of oscillation t' observed again.

The required moment of inertia of the body alone is then—

$$K = K' \frac{t^2}{t'^2 - t^2}$$

This method is specially applicable to bodies hung by a thread, so as to turn about a vertical axis, particularly therefore to magnets. The known moment of inertia may be added by weighting the magnet with a ring of known dimensions and weight, or by hanging two similar cylindrical weights upon points, or by threads, at equal distances from the axis of revolution (the suspending thread), and so that the axes of the cylinders are vertical. The turning force is unaltered by the added weight, as only the horizontal force is taken into consideration. The moment of inertia of the two cylindrical weights together is—

$$K = m(l^2 + \tfrac{1}{2}r^2)$$

m being the mass of both together, l the horizontal distance of the centres of suspension (points or threads) of the weights from that of the magnet (its axis of revolution), and r the radius of the cylinders.

The expression assumes that the weights turn with the magnet, being attached either bifilarly, suspended on points with great friction, or fixed pins. If they were suspended by quite thin threads, so that they did not turn perceptibly on their axes with the motion of the magnet, we should substitute $K' = ml^2$. l is determined by measuring the whole distance between the points of suspension of the weights and halving it. In bifilar suspensions, the distance of the two threads on each side is measured, and the mean taken as $2l$. Fixed weights may have turned circles from which the distance of their centres may be measured. Eccentricities of the centres of gravity are compensated by turning the weights 180°. For further details on the general question of the co-oscillation of attached loads, see Kreichgauer, *Wied. Ann.* xxv. p. 289, 1885.

Example.—

Diameter of cylinders 1·00 cm. .	$r =$ 0·50 cm.
Their combined weight 50·00 grm. .	$m =$ 50·00 grm.
The measured distance apart of their axes = 10·026 cm. . . .	$l =$ 5·013 cm.

$$K' = 50{\cdot}00\left(5{\cdot}013^2 + \frac{0{\cdot}25}{2}\right) = 1262{\cdot}8\ [\text{g. cm.}^2]\ (\text{or } 126280000\ [\text{mg. mm.}^2])$$

Further, the periods of oscillation are found to be—of the unloaded magnet, $t = 9{\cdot}737$ sec.; of the loaded, $t' = 14{\cdot}267$ sec.

The required moment of inertia is—

$$K = 1262{\cdot}8 \frac{9{\cdot}737^2}{14{\cdot}267^2 - 9{\cdot}737^2} = 1101{\cdot}1\ [\text{g. cm.}^2]$$

III. *By Bifilar Suspension* (F. Kohlrausch).—A bifilar suspension is assumed in which the body can be laid. From its weight and the dimensions of the threads, the directive force D_0 is calculated according to **53.** The time of oscillation being t_0, the moment of inertia of the suspension (end of **53**), is

$$K_0 = \frac{D_0 t_0^2}{\pi^2}$$

The body of which the moment of inertia K is required is next laid in the suspension, so that its centre of gravity lies in the mean vertical of the threads. Calling the directive force now D and the period of oscillation t_1—

$$K = \frac{1}{\pi^2}(Dt^2 - D_0 t_0^2)$$

K is referred to a vertical axis passing through the centre of gravity.

On the observation of rapid oscillations, compare p. 224.

If the body to be determined is already magnetised it may be observed in two opposite meridional positions. If t_1 and t_2 are the respective periods of oscillation—

$$t^2 = \frac{2t_1^2 t_2^2}{t_1^2 + t_2^2}$$

55.—Coefficient of Torsion of a Suspended Magnet (Gauss)

The moment of torsion of the thread and the moment of rotation of terrestrial magnetism are proportional to each other for small deflections. The relation of the former to the latter is called the ratio or coefficient of torsion, and is measured in the following manner:—The position of the magnet is first observed; then by turning the upper or lower points of attach-

ment of the thread, a measured torsion a is communicated to it, and the position of the magnet is again observed. Calling $\phi =$ the angle through which the torsion deflects the magnet, the required coefficient of torsion Θ is—

$$\Theta = \frac{\phi}{a - \phi}$$

In instruments for fine measurements the suspending fibre is attached, either above or below, to a graduated circle, by turning which any degree of torsion may be produced. The angle of rotation read on this circle is a. In the absence of such a circle the magnet must be turned once entirely round without moving the upper attachment of the thread; a will then be 360°.

The deflection is measured with mirror and scale. If this be e, with the scale at the distance A, the angle $\phi = 57°{\cdot}3 \times \dfrac{e}{2A}$. If a be a whole rotation, the calculation may be simplified by writing $a = 2\pi = 6{\cdot}28$, and $\phi = \dfrac{e}{2A}$.

The moment of torsion of the suspension may also be determined independently, by hanging to the thread a mass of known moment of inertia (K **54**, I.), and observing the period t of its oscillations. The moment of torsion is then, in absolute measure, $d = \pi^2 k/t^2$ (App. 10). If, at the same time, we know the directive force of the magnet to be suspended, for instance from the magnetic force of the bar M, and that of terrestrial magnetism H, as $D = MH$ (**62**, or App. 16), the coefficient of torsion is $\Theta = d/(D + d)$.

The lighter the magnet, the smaller may we make the coefficient of torsion, since the tensile strength increases as the square, but the moment of torsion only as the fourth power of its thickness. Cocoon fibres have very variable moments of torsion according to their origin. Fine fibres of 10 cm. long, from the interior of a cocoon, may have moments of torsion as low as $d = 0{\cdot}0001$ cm. g., so that frequently their coefficient of torsion may be neglected; while others reach ten times that amount, and their tensile strength is also very variable.

MAGNETISM.

55A.—General.

The maximum permanent magnetism of a steel bar depends on its mass, dimensions, and hardness. Slender form is favourable to high magnetic moment; and in general, for constrained magnets great hardness is desirable. With regard to qualities and hardness of steel, compare Holborn, *Zeits. f. Instr.* 1891, 114.

The magnetic moment, divided by the mass of the bar, is known as its specific magnetism. Its extreme limit, which however cannot permanently be maintained, is about 200 [cm. g.] (App. 15) per grm. of iron (v. Waltenhofen). In magnets of very elongated form, a permanent magnetism of about 100 may be reached, but in ordinary bars it does not often exceed 40.

A freshly magnetised bar loses a part of its magnetism, at first rapidly, but afterwards more slowly. Long boiling facilitates the attainment of a permanent condition. After first magnetising, the bar is boiled for some time, re-magnetised, and the boiling repeated, and so on; the final boiling being continued for six hours or more. Magnets thus treated are much more constant than the ordinary ones (Strouhal and Barus, *Wied. Ann.* xx. p. 662, 1883).

Separation of Poles.—For magnetic action at a distance, we may suppose the two magnetisms of an ordinary magnet to be concentrated in two points or poles, which are separated on the average by about $\frac{5}{6}$ of the length of the bar. (Compare App. 15 and **62B**.)

Suspension of Magnets.—Where a long suspension, as for

instance from the ceiling, is practicable, the larger magnets are best suspended by hard brass wire, which combines great tensile strength with a moderate modulus of elasticity (Table 17). In other cases, cocoon fibres (end of **55**), or bundles of them, must be employed. These may be formed by winding a long fibre round two glass rods fixed at the required distance apart, on the edge of the table. After winding the necessary number of turns, the ends of the fibre are knotted together, the tension is equalised as much as may be, and the outer end of the bundle is looped round the upper or lower hook of the suspension, the tension being again equalised as far as possible before tightening the loop.

Fig. 38.

Single fibres are looped as in the annexed figure, and are finally drawn tight, but not so much so that the hanging thread cannot still be drawn up. Hanging in loose loops is to be avoided. Free ends of fibres must be cut close off to avoid friction.

Fig. 39.

Variations of Terrestrial Magnetism.—The changeableness of terrestrial magnetism may at times seriously interfere with observations. Ordinarily these are at a minimum about noon, but such disturbances may occur at any hour.

Astatisation of Magnetic Needles.—Occasionally, and especially for galvanometric work, a diminution of the earth's directive force is desirable. For this purpose pairs of needles with opposing poles may be employed; the needle may be surrounded with a "guard ring" of soft iron, which, by its own magnetisation, neutralises that of the earth; the needle may be hung bifilarly in the reverse position to that it naturally assumes; or, finally, a compensating magnet is fixed in a suitable position, not too near the needle, and exerting on it a magnetic force opposing that of the earth. In both the latter cases the effect of variations of declination are of course increased. It is needless to point out that similar means can be employed either to strengthen the directive force, or to give the needle another azimuth than that of north and south.

Instruments for Terrestrial Magnetism.—Compendious instruments for the observation of declination, and in some cases

for that of intensity also, have been contrived by Fox, Lamont, Meyerstein, Neumayer, Weber, Wild, and others.

56.—Magnetic Inclination.

Inclination is the angle which the direction of terrestrial magnetic force makes with the horizontal (Table 24).

The placing of the divided circle in the magnetic meridian is accomplished by the aid of an ordinary compass-needle, for which an accuracy within 1° is sufficient.

The numbering of the divisions of the circle varies in different instruments. It is convenient when in each quadrant the divisions are numbered from the horizontal as zero; and for simplicity we will suppose, in the following, that this is the case.

An inclination instrument with fixed circle is first placed vertical by a plummet hung from the uppermost division of the circle. In an instrument with rotating circle the axis of rotation must be made vertical, which is shown to be the case by the bubble of a spirit-level applied to the instrument taking the same place in all positions of the circle. A systematic method is described under **88**, 1.

In each position of the needle, both points must be read and the mean taken.

On account of possible lateral eccentricity of the centre of gravity, the needle must now be turned round (with a movable circle, the circle and needle together must be turned 180°), by which we also eliminate the deviation of the geometric from the magnetic axis of the needle (and, with movable circles, any deviation of the line joining the upper and lower 90° divisions from the axis of rotation of the instrument). Any longitudinal displacement of the position of the centre of gravity requires for its elimination a reversal of the magnetism of the needle.

We must observe the angles—

(1.) ϕ_1 in the first position of the needle.
(2.) ψ_1 when the needle is turned 180° round its magnetic axis, and again replaced in the instrument; or, with movable circle, when the latter, with the needle, is turned 180°.

(3.) ϕ_2 when the magnetism of the needle is reversed by stroking with a bar magnet in position 1.

(4.) ψ_2 when the re-magnetised needle is placed in position 2, or the circle turned 180°.

I. If these angles are nearly alike, the inclination i is the arithmetical mean—

$$i = \tfrac{1}{4}(\phi_1 + \psi_1 + \phi_2 + \psi_2)$$

II. In any case it may easily be managed by grinding the side of the needle before the observation that ϕ_1 and ψ_1, and also ϕ_2 and ψ_2, are nearly alike; and then—

$$\tan i = \tfrac{1}{2}[\tan \tfrac{1}{2}(\phi_1 + \psi_1) + \tan \tfrac{1}{2}(\phi_2 + \psi_2)].$$

III. Should, however, ϕ_1 and ψ_1 also differ considerably, we must write—

$$\cot a_1 = \tfrac{1}{2}(\cot \phi_1 + \cot \psi_1)$$
$$\cot a_2 = \tfrac{1}{2}(\cot \phi_2 + \cot \psi_2)$$

and calculate lastly—

$$\tan i = \tfrac{1}{2}(\tan a_1 + \tan a_2)$$

Formulæ II. and III. are obtained by supposing the unknown displacement of the centre of gravity resolved into its components, parallel and perpendicular to the magnetic axis of the needle, and considering the conditions of equilibrium between magnetic force and that of gravitation.

Were there, for instance, only a longitudinal displacement l of the centre of gravity towards the north pole of the needle, then taking ϕ_1 the observed angle, p the weight of the needle, M its magnetic movement, and C the total intensity of terrestrial magnetism (**59**, and App. 16), we should have—

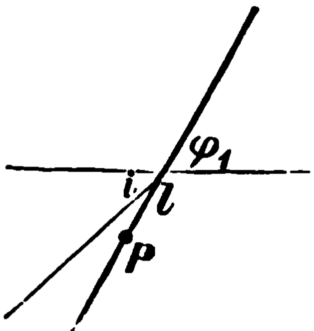


Fig. 40.

$$pl \cos \phi_1 = MC \sin (\phi_1 - i)$$

If now the magnetisation of the needle be reversed, so that the displacement of the centre of gravity is towards the southern end, we have—

$$pl \cos \phi_2 = MC \sin (i - \phi_2)$$

By cross multiplication of these two equations, and elimin-

ation of the sines by division by $\cos i \cos \phi_1 \cos \phi_2$, we obtain—

$$\tan i - \tan \phi_2 = \tan \phi_1 - \tan i$$

from which II. follows. III. is deduced similarly.

It is assumed that the magnetic moment of the needle is the same before and after remagnetisation, which is very nearly the case if it be performed by carefully and equally stroking a thin needle. It is, in any case, advisable that the displacement of the centre of gravity should not give rise to too great differences of position before and after remagnetisation.

The stroking itself is performed in the following manner: —Holding the needle by one half, with the fingers near the axis of rotation, the other half is drawn lengthwise completely over the pole of a magnet, as in the figure. So, for instance, the two surfaces of one end should be twice gone over; then those of the other end four times, and then the first twice again.

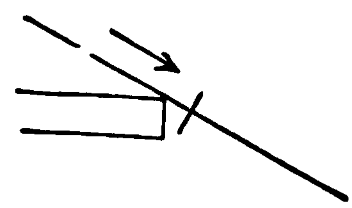


Fig. 41.

On account of the friction it is well to deduce the position of rest of the needle from observations of oscillation (**8**).

For complete directions, see Gauss's Works, vol. v. p. 444.

57.—Declination of Terrestrial Magnetism.

By "declination" is understood the angle which the magnet makes with the astronomical meridian; and to indicate the direction of the deflection, the angle is counted from the latter to the former. With us, therefore, the declination is "west." As we cannot be certain of the position of the magnetic axis of a needle, we must, for exact determination, observe the magnet in two positions.

For the measurement (after Gauss) we require a theodolite with a horizontal circle, and a sight-mark, of which the astronomical azimuth from the place of the theodolite is known; and which may be either cross-wires in the focus of a lens in the observatory, or a distant terrestrial mark, the azimuth of which has been fixed by the aid of the pole-star, or of the sun (compare **88** and **89**); and lastly, a magneto-

meter, of which the needle can be turned 180° on its axis. The theodolite is placed nearly in the same magnetic meridian as the suspending thread of the needle, and its telescope at the same height as the magnet.

We assume, as is most convenient, that the magnet has a longitudinal sight, which at the end towards the theodolite has a lens of the same focal length as the length of magnet. At the other end is a mark (screen with small opening, cross-threads, or divided glass), which, seen through the lens, appears as a distant object. A mirror attached to the magnet, of which the normal is nearly coincident with the magnetic axis, will answer the same purpose if the cross-wires of the theodolite can be illuminated; the telescope being focussed on the reflection of its cross-wires.

The theodolite is divided so that the number increases in turning the telescope the same way as the sun (that is, from left to right).

After adjusting vertically the axis of rotation of the theodolite, the telescope is pointed so that the terrestrial mark appears on the cross-wires. Let the reading of the circle be now $= a$. If Z be the astronomical azimuth of the mark, counting from north to west (see previous page), the theodolite must be turned to the division $a + Z$ in order that the line of vision of the telescope may point north.

The telescope being directed to the mark on the magnet, let the reading of the circle be a_1.

The magnet is turned on its axis 180°, or so that the side is uppermost which was previously below, and the telescope directed again to its mark. Let the reading of the circle be a_2. The readings a_2 and a_1 always differ but very slightly.

Now clearly the westerly declination will be—

$$\delta' = a + Z - \frac{a_1 + a_2}{2}$$

when the suspending thread has no torsion. To determine and eliminate the latter, we must measure the angle to which the thread has been twisted in the observation. For this purpose the magnet must be taken from its stirrup, an unmagnetised bar of equal weight substituted for it, and the

turning of the stirrup by this change measured on a divided circle laid underneath. Should this angle of rotation $=\phi$ in the same direction as the sun's daily course, the declination will be—

$$\delta = \delta' + \Theta\phi$$

Θ being the ratio of torsion (**55**).

Variations.—To measure variations of declination, a magnetometer is employed which consists of a suspended magnet provided with a mirror, and a fixed telescope and scale (**48**). If A be the distance of the scale from the mirror measured in scale divisions, and Θ the rates of torsion (**55**), a scale division has the value in absolute angular measure (App. 3) of $(1+\Theta)/2A$; or, in minutes of arc, of 1719 $(1+\Theta)/A$ (**49**). On the observation of oscillating needles see **50**.

58.—Surveying with the Compass.

Table 23 contains the angles of deviation of the magnetic from the astronomic meridian, for the (geographical) latitudes and longitudes of Mid-Europe. The declination so obtained will never, out of doors, differ from the actual more than $\frac{1}{2}°$. This possibility of determining an astronomical direction with the magnetic needle is of the greatest value in surveys where only moderate accuracy is required. Table 23A gives similar information for N. America, but probably with somewhat less accuracy.

On the use of the instruments concerned we will not touch further than to say that the universal directions for instruments for angular measurement are applicable to them. The accuracy is principally dependent on the length of the compass-needle, since the shorter it is the greater is the possible difference between its magnetic and geometric axes.

The influence of friction on the point is lessened by slightly shaking the compass before reading. It is obvious that both ends of the needle should always be read.

It is perhaps not superfluous to note, that a compass on any stand capable of rotation, may take the place of a divided circle, the ends of the needle serving as indices.

59.—Measurement of Horizontal Intensity of the Earth's Magnetism (Gauss).

The intensity of the magnetic force at any place, or the strength of a magnetic field, is the force which it exerts on a unit magnetic pole. The unit pole again is defined as exerting on a similar pole at unit distance a unit force. (Compare App. 14 to 16.)

The measurement depends on two observations — viz. of a *time* of oscillation and of an *angle* of deflection. From the first may be obtained the product $P = MT$, of the horizontal intensity T of the earth's magnetism, and the magnetic moment M of the swinging magnet, if the moment of inertia of the magnet be known. The ratio $Q = \frac{M}{T}$ is found by observing the deflection of another magnetic needle, caused by bringing the first to a measured distance from it. From these two numbers M may be eliminated by division and T determined.

Gauss reckoned all times in seconds, lengths in millimeters, and masses in milligrams. According to present usage, the numerical example is given in centimeters and grams, which yield a number for the earth's magnetism one-tenth of Gauss's. (See Appendix 14 to 16, and Table 28.)

I. *Determination of MT by Oscillation.*

The period of oscillation of the magnet, suspended by a thread (**55A**), and swinging in a horizontal plane, is determined. If, then,

t = the period of oscillation in seconds, and reduced to an infinitely small arc (**52**);
K = the moment of inertia of the magnet (**54**);
Θ = the ratio of torsion of the thread (**55**);

the required product—

$$MT = \frac{\pi^2 K}{t^2(1+\Theta)} = P$$

For the directing force acting on the magnet is $MT(1+\Theta)$,

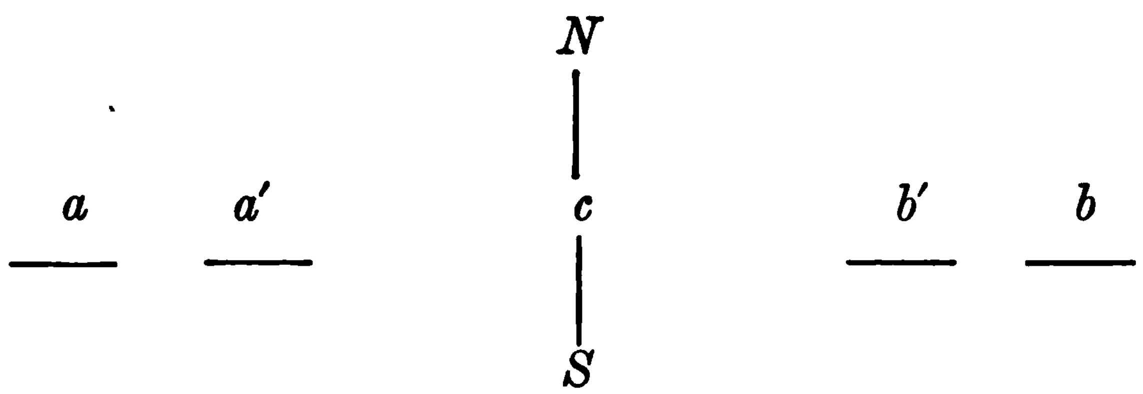
and the square of a time of oscillation divided by π^2 gives the ratio of the moment of inertia to the directing force (Appendix 10). On the determination of MT with the balance compare III.

II. *Determination of* $\frac{M}{T}$ *by Deflection.*

By allowing the magnet used above to act from two pairs of equal distances on a magnet needle which can oscillate in a horizontal plane, and each time observing the angle through which the needle is deflected, we obtain the ratio of the magnetic moment M to the horizontal force of the earth's magnetism according to the following rules. (For the effect of torsion of the suspending thread, see pp. 232, 243.)

FIRST POSITION—

c is the centre of the compass :—



The line NS represents the magnetic meridian, *i.e.* the position which the free needle takes. The deflecting magnet is placed east or west of the compass-needle, and on the same level, so that its centre is in the positions a, a', b', b, successively; the distances of the centre of the magnet from that of the compass are equal in pairs, $ac = bc$, $a'c = b'c$.

The bar is placed, for instance, at a, with the north pole westward. The position of the compass-needle is read off at both ends. The deflecting bar is turned round 180°, so that its opposite pole is towards the compass-needle, which is now deflected in the opposite direction, and again read at both ends. The differences of the two positions of each end are halved, and the arithmetical mean of the two halves taken as the angle of deflection for the position a of the deflecting magnet.

It is supposed in the foregoing that the circle of the com-

pass is divided in one direction from 0° to 360°, the most convenient arrangement. If, as is sometimes the case, there are two zeros, from each of which it is numbered to both sides, instead of the half differences of the readings, we must, of course, take their half sum. Exactly the same is done for the positions *a'*, *b'*, and *b*. The arithmetical means of the nearly equal angles for *ab*, and for *a'b'*, must then be taken (each will thus be the result of 8 single observations).

If we take

ϕ = the mean angle of deflection for *a* and *b*;
ϕ' = „ „ *a'* and *b'*;
r = half the distance *ab* in millimeters;
r' = „ *a'b'* „

then the required number—

$$\frac{M}{T} = \frac{1}{2}\frac{r^5 \tan\phi - r'^5 \tan\phi'}{r^2 - r'^2} = Q$$

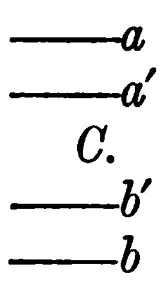
The required horizontal intensity or strength of field is then

$$T = \sqrt{\frac{P}{Q}}$$

Proof for a Short Needle.—If a short needle, lying in a line with the magnetic axis of a bar magnet of magnetic moment *M*, with poles east and west, at a sufficient distance *r* from the centre of the needle, be deflected through the angle ϕ, then, according to Gauss (compare Appendix 15 and 16), $\tan\phi = \frac{2}{r^3}\frac{M}{T}\left(1 + \frac{\eta}{r^2}\right)$, where η is a constant for each magnet. This, combined with the corresponding expression for the distance r', permits of the elimination of the constant η, and we obtain $r^5 \tan\phi - r'^5 \tan\phi' = 2\frac{M}{T}(r^2 - r'^2)$.

SECOND POSITION—

$\frac{M}{T}$ may also be obtained by placing the deflecting magnet, as in the annexed diagram, at equal distances successively *north* and *south* of the compass *C.*, and obtaining two pairs of such observations for different distances as before. The same mode of procedure must be followed as has been previously described, both in regard to the observation and in calculating the mean value. Using also the same notation as before for the distances of the centre of the



deflecting magnet from the compass—viz. $r=\frac{1}{2}\ ab$, $r'=\frac{1}{2}\ a'b'$; and further, taking ϕ and ϕ' for the mean angles of deflection for the positions *ab* and *a'b'*, then according to Gauss,

$$\frac{M}{T}=\frac{r^5 \tan\phi - r'^5 \tan\phi'}{r^2 - r'^2}$$

By reading the angle of deflection from both ends of the needle, and taking the mean, the influence of any eccentricity of its centre with regard to the graduated circle of the compass disappears. The poles of the deflecting magnet are reversed, to eliminate the effect of any unsymmetrical magnetisation in itself. The same thing is accomplished for the compass-needle by causing the deflections alternately to each side. It is obvious that, at the same time, the exactness of the results will be increased in proportion to the eightfold repetition of each single reading.

Favourable Distances.—In order that the errors of observation may have the least possible influence on the result, it is best that the ratio of the two distances r/r' should equal 1·4. The angles of deflection should be sufficiently large, but to produce this the deflecting bar must not be brought so near the needle as to make *the lesser distance* r' *less than* 4 *times the length of the bar,* since otherwise, in addition to the term $\frac{\eta}{r^2}$ (*v.s.*) a second with $\frac{1}{r^4}$ of material value, would have to be added. At the above distance, however, the deflections of a compass with divided circle are too small to allow of any high degree of accuracy.

Reading by Mirror.—Should the deflections be measured by a magnetometer with mirror and scale (**40**, **49**), its coefficient of torsion ϑ (**55**) must be taken into account by multiplication of the tangent by $1+\vartheta$. At the same time, variations in declination must be eliminated by suitable change of the deflections, or the simultaneous observation of a variometer.

Simplification when the same Magnet is repeatedly used.—The deflection at two different distances is necessary for the elimination of the unknown distribution of magnetism in the bar and the needle, which is accomplished by the

foregoing formula. If the same bar and needle be repeatedly used for the determination of T, the observation and calculation may be simplified. It is only necessary, once for all, to make the observation for two different distances. From this is calculated the factor η—

$$\eta = r^2 r'^2 \frac{r^3 \tan\phi - r'^3 \tan\phi'}{r'^5 \tan\phi' - r^5 \tan\phi}$$

If, then, the angle of deviation Φ be found for one suitable distance R of the bar, we have simply

$$\frac{M}{T} = \tfrac{1}{2}\,\frac{R^3 \tan\Phi}{1 + \frac{\eta}{R^2}};$$

or, similarly, omitting the factor $\frac{1}{2}$, by the 2nd method (p. 242).

Simplification by Introduction of Polar Separation.—The magnetism of slender bars may be considered in regard to action at a distance, as being concentrated in two points called poles or distance-poles. In ordinary magnets these poles are about $\frac{1}{12}$ of the length of the bar from the two ends, and the polar separation is about $\frac{5}{6}$ of the total length. If this be denoted for the bar by $\mathfrak{L}$ and for the needle by $\mathfrak{l}$, the correction-factor η for small deflections (**62**B, and App. 15), will be,—

in the first arrangement $\eta = \frac{1}{2}\mathfrak{L}^2 - \frac{3}{4}\mathfrak{l}^2$
in the second arrangement $\eta = \frac{3}{8}\mathfrak{L}^2 + \frac{3}{2}\mathfrak{l}^2$.

Changes in Gauss's Formula.—For short needles, the following formulæ are more exact than those previously given, and especially so for short distances of the magnets.

First Arrangement

$$\frac{M}{T} = \tfrac{1}{2}\left[\frac{r^2 - r'^2}{r^{\frac{1}{2}} \tan\phi^{-\frac{1}{2}} - r'^{\frac{1}{2}} \tan\phi'^{-\frac{1}{2}}}\right]^2$$

Second Arrangement

$$\frac{M}{T} = \left[\frac{r^2 - r'^2}{\tan\phi^{-\frac{2}{3}} - \tan\phi'^{-\frac{2}{3}}}\right]^{\frac{3}{2}}$$

or by observation from one distance R only:

$$\frac{M}{T} = \tfrac{1}{2}R^3 \tan\Phi\left(1 - \tfrac{1}{4}\frac{\mathfrak{L}^2 - \frac{3}{2}\mathfrak{l}^2}{R^2}\right)^2$$

$$\frac{M}{T} = R^3 \tan\Phi\left(1 + \tfrac{1}{4}\,\frac{\mathfrak{L}^2 - 4\mathfrak{l}^2}{R^2}\right)^{\frac{3}{2}}$$

Correction on Account of Induced Magnetism in the Bar.
—During the oscillations, the magnet lies north and south, and its magnetism is therefore somewhat strengthened by the earth's induction. It amounts therefore to $M(1+\Delta)$ where Δ denotes the induction-coefficient of the horizontal component of the earth's magnetism. The value P therefore, which has been determined (p. 240) represents, not MT simply, but $MT(1+\Delta)$, and therefore we have not $T=\sqrt{P/Q}$, but

$$T=\sqrt{\frac{P}{Q}}\sqrt{\frac{1}{1+\Delta}}$$

The correction for T on account of induced magnetism is therefore $\sqrt{1/(1+\Delta)}$, for which (Formula 6, p. 11) we may write

$$1-\tfrac{1}{2}\Delta$$

On the measurement of Δ see **81A**. For ordinary magnets, Δ may be approximately estimated by the rule that the magnetic field 1[cm. g.] induces approximately in 1 grm. steel, a magnetism of 0·25 [cm. g.]. If therefore the magnet weighs p grm. and the terrestrial magnetism is T, $M\Delta=0{\cdot}25\,pT$, or $\Delta=0{\cdot}25\,pT/M$.

The observations of oscillation and deflection should, of course, be made in the same place. Iron articles, which might exercise a local influence (and especially articles in the pocket of the observer, or steel spectacles, and notebooks bound with iron wire), must be removed from the neighbourhood. Variations of magnetism of the earth or of the bar (the latter specially through change of temperature) are most likely to be excluded when the two sets of observations follow each other as closely as possible. As regards accurate corrections compare **61** and **62A**.

Example—

(I.) *Determination of MT.*

Moment of Inertia.—The magnetic bar is a right-angled parallelopiped, of which the length $a=10{\cdot}00$ cm., and the breadth $b=1{\cdot}25$ cm. Its weight $m=119{\cdot}86$ grm. By (**54**, p. 228) its moment of inertia—

$$K=119{\cdot}86(10{\cdot}00^2+1{\cdot}25^2/12=1014{\cdot}4\ cm.^2g.$$

Ratio of Torsion of Suspending Thread.—It was found that a single complete rotation of the thread produced a deflection of the magnet of 1°·4. By **55** the ratio of torsion—

$$\Theta = \frac{1 \cdot 4}{360 - 1 \cdot 4} = 0 \cdot 0039$$

Time of Oscillation.—This was found to be (**52**) 7·414 sec., where the mean arc of oscillation was 30°. This, reduced to an infinitely small arc, gives for time of oscillation—

$$t = 7 \cdot 414 - 7 \cdot 414 \times 0 \cdot 0043 = 7 \cdot 382 \textit{ sec.}$$

Calculation of MT.—The required value is—

$$MT \frac{\pi^2 K}{t^2(1+\Theta)} = \frac{3 \cdot 1416^2 \times 1014 \cdot 4}{7 \cdot 382^2 \times 1 \cdot 0039} = 183 \cdot 01 \textit{ cm.}^2\textit{g./sec.}^2$$

(II.) *Determination of* $\frac{M}{T}$

A compass stands on the 50th division of a rule divided into centimeters, and lying east and west. The same magnet as was used in the determination of *MT* is placed with its centre successively on the 10th, 20th, 80th, and 90th divisions, twice on each division—once with its north and once with its south pole towards the compass (see Fig. on p. 241). In these positions the following observations of the needle are made :—

When the magnet was placed on 100 the readings are—

	1st end	2nd end
N pole towards needle	99°·4	279°·8
S pole towards needle	79°·9	260°·6
Half difference	9°·75	9°·60
Mean	9°·67	

In a similar manner is found, when the centre of magnet lies—

at 20 *cm.*	22°·41	The means are therefore—
„ 80 „	22°·67	$\phi = 22°\cdot54$ for $r = 30$ *cm.*
„ 90 „	9°·87	$\phi' = 9°\cdot77$ „ $r' = 40$ *cm.*

Therefore—

$$\frac{M}{T} = \frac{1}{2}\,\frac{40^5 \tan 9°\cdot77 - 30^5 \tan 22°\cdot54}{40^2 - 30^2} = 5388 \textit{ cm.}^3$$

and

$$T = \sqrt{\frac{183 \cdot 01}{5388}} = 0 \cdot 1843 \frac{g^{\frac{1}{2}}}{\textit{cm.}^{\frac{1}{2}}\textit{sec.}}$$

The expression η (p. 242) is, according to these experiments,

$$\eta = 40^2 \times 30^2 \frac{30^3 . \tan 22^\circ{\cdot}54 - 40^3 . \tan 9^\circ{\cdot}77}{40^5 . \tan 9^\circ{\cdot}77 - 30^5 . \tan 22^\circ{\cdot}54} = 36{\cdot}3 \; cm.^2$$

In fact, calculation by the formula

$$\frac{M}{T} = \tfrac{1}{2} \frac{30^3 . \tan 22^\circ{\cdot}54}{1 + 36{\cdot}3/900} \quad \text{or} = \tfrac{1}{2} \frac{40^3 . \tan 9^\circ{\cdot}77}{1 + 36{\cdot}3/1600}$$

results in the same value 5388.

Calculation with Polar Separation.—The length of magnet $= 10{\cdot}0$ cm., that of needle $= 2{\cdot}0$ cm., therefore the two polar distances are

$$\mathfrak{L} = \tfrac{5}{6} \times 10 = 8{\cdot}3 \; cm. \qquad \mathfrak{l} = \tfrac{5}{6} \times 2{\cdot}0 = 1{\cdot}7 \; cm.$$

If we suppose that we have only a single observation, viz. that of the angle of deflection $\Phi = 9^\circ{\cdot}77$ at the distance $R = 40$ *cm.*, then (p. 244)

$$\frac{M}{T} = \tfrac{1}{2} R^3 \tan \Phi \left(1 - \tfrac{1}{4} \frac{\mathfrak{L}^2 - \frac{3}{2}\mathfrak{l}^2}{R^2}\right)^2 = 5510(1 - 0{\cdot}0101)^2 = 5399$$

In order not to be obliged to calculate into minutes the fractional parts of the degrees as read off, we may make use of the excellent "five figure" tables of Bremiker.

III. *Determination of MT with the Balance* (Töpler).

A delicate balance, free from iron, is capable of rotation on a vertical axis. The beam is in the magnetic meridian, and with it the magnet M is rigidly connected in a vertical position; the moment of rotation exerted on the beam by the earth's horizontal magnetic force acting on M is $M . T$. If the entire balance is turned through 180°, the same moment of rotation acts in the opposite direction. Different weights will therefore be required to produce equilibrium in the two positions.

If this difference be m *grm.* and l *cm.* be the length of the balance arm, and lastly $g = 981$ *cm./sec.*^{2}, the acceleration of gravity, clearly

$$MH = \tfrac{1}{2} \, gml \; [g \, . \, cm.^2/sec.^2]$$

Compare Töpler, *Wied. Ann.* xxi. 158, 1884; Freyberg, *ibid.* xxv. 511, 1885.

59A.—THE MAGNETIC THEODOLITE.

A magnetic theodolite (Lamont, Meyerstein, Neumayer) is a combined instrument for the determination of both declination and horizontal intensity. In the axis of rotation is placed the magnetometer, while the telescope is attached outside, as in the spectrometer. On determination of declination compare **57.**

The measurement of intensity involves, as in **59**, firstly, the observation of period of oscillation and moment of inertia of a magnet; and, secondly, that of the deflection of a needle. The angle of deflection is observed with the telescope of the theodolite itself, the reflection of the illuminated cross-wires of which in a mirror attached to the needle (**39**, 2) or a sight on the latter (compare **57**) serving for exact adjustment.

In the much-used Lamont's theodolite, the telescope turns together with the magnetometer and the bars on which the deflecting magnet is laid. Hence the needle at the moment of observation is brought perpendicular to the line between it and the deflecting magnet, and instead of the tangent, the sine of the angles must be taken, the calculation being made by the formula

$$\frac{M}{T}\left(1+\frac{\eta}{r^2}\right)=\tfrac{1}{2}r^3 \sin\phi$$

The second term of the correction with $1/r^4$, which otherwise might be of influence, must be eliminated by making the needle 2·1 times smaller than the magnet, in which case their lengths mutually nearly compensate each other.

The quantity η is determined, once for all, as in **59**, p. 242, by observations at two distances. As there described, the needle is deflected both from east and west, and each time in two positions of the magnet. The corrections for induced magnetism, and for want of symmetry, here need merely be noted.

A considerably improved form of magnetic theodolite, as regards transport and convenience, has been constructed by Neumayer. The needle, which is observed by an attached mirror, is reversible, but moves on a point. The thread-suspension is only employed in observing the oscillation.

S. Eschenhagen in Kirchhoff's *Anleitung zur Deutsches Landes- und Volksforschung*, p. 118.

Note.—As regards the form of magnetometer used by the English Government, which is similar to that of Lamont, above described, see *Admiralty Manual of Scientific Inquiry*, and *Airy on Magnetism*, p. 57—*Trans.*

60.—Determination of Horizontal Intensity by the Compensated Magnetometer (W. Weber).

The compensated magnetometer is principally intended for the comparison of horizontal intensity at different places, but will also serve for absolute determination. It consists of a compass and a frame carrying four magnets of similar form to the compass-needle. The two smaller of these are twice the length, breadth, and thickness of the compass-needle, while the larger are threefold. When the frame is placed with its four holes on corresponding pins on the compass, the smaller bars are east and west of the needle (p. 241), and the larger ones north and south (p. 242). The deflecting force of all the bars must act in the same direction, and therefore the poles of the smaller magnets must be in the opposite direction to those of the larger ones. The distance between the larger bars should be about 1·20 times that of the smaller ones.

Observation of Deflection.—The compass is so placed that when the frame is set upon it the line connecting the larger magnets is north and south. The frame being put on, the position of the needle is observed, the frame is turned 180° in its plane, and the position again noted, both ends of the needle being read each time. The half difference of these two positions is the angle of deflection ϕ.

Observation of Period of Oscillation.—A small pin is screwed into one of the holes near the large magnets, and by this the frame is hung in a stirrup attached to a cocoon thread. A mirror may also be screwed into another hole near the point of suspension for observation with telescope and scale. To determine the moment of inertia, two cylindrical weights are employed, which are hung by cocoon threads over the outer end-surfaces of the frame.

I. *Comparison of Horizontal Intensity at Two Places.*

The horizontal intensities of the places are inversely proportional to the tangents of the angles of deflection—

$$\frac{T_1}{T_2}=\frac{\tan\phi_2}{\tan\phi_1}$$

Differences of temperature may be calculated for, provided "the temperature coefficient" of the magnets is known (**62** A).

If, however, the magnetism of the bars be altered, we must also observe the times of oscillation, t_1 and t_2, of the frame in the two places, when all four magnets have their poles in the same direction. Then

$$\frac{T_1}{T_2}=\frac{t_2}{t_1}\sqrt{\frac{\tan\phi_2}{\tan\phi_1}}$$

II. *Determination of Absolute Horizontal Intensity.*

If we call

$2r$ the distance of the centres of the smaller (east and west) magnets from each other;
$2R$ that of the larger magnets;
ϕ the angle of deflection;
t the time of oscillation with magnets all in the same direction;
τ that when the smaller magnets are turned 180°;
Θ the ratio of torsion of the thread in the first case;
K the moment of inertia.

We then have the absolute horizontal force—

$$T=\frac{\pi}{t\,\tau}\sqrt{\frac{K}{\tan\phi}\left(\frac{\tau^2-t^2}{r^3}+\frac{\tau^2(1-2\,\Theta)+t^2}{2R^3}\right)}$$

Compare also *Pogg. Ann.* Bd. 142, S. 551.

60A.—Determination of Horizontal Intensity by the Bifilar Method (F. Kohlrausch).

I. *Determination of MT. Absolute Bifilar Magnetometer.*

The stirrup of a bifilar suspension is directed east and west. A magnetic bar is placed in it, and the scale is then

read. The position of the magnet is reversed, end for end, and the scale is read again. Half the angle between the two positions is $= a$ (**48**, **49**).

If the directive force of the bifilar suspension, determined after **53** $= D$; the terrestrial magnetism T, and that of the bar M,

$$MT = D \,.\, \tan a$$

II. *Determination of M/T.*

The above magnet, directed east and west, deflects a short magnetometer needle, which is situated north or south of the magnet, as in the second arrangement at the considerable distance r, through the angle ϕ; Θ being the coefficient of torsion of the needle (**55**), and $\mathfrak{L}$ the separation of poles of the bar (that is $\frac{5}{6}$ of its length; p. 244, and **62**B). Then

$$\frac{M}{T} = r^3\left(1 + \frac{3}{8}\frac{\mathfrak{L}^2}{r^2}\right)(1 + \Theta) \tan \phi$$

By multiplying the two equations, M is obtained; division yields

$$T^2 = \frac{D}{r^3(1 + \Theta)\left(1 + \frac{3}{8}\frac{\mathfrak{L}^2}{r^2}\right)} \frac{\tan a}{\tan \phi}$$

The effect of magnetic variations in both earth and bar will be avoided if a and ϕ are observed simultaneously, that is, if the bar deflects the magnetometer while it is itself bifilarly suspended. The observation is made with the magnetometer placed to the north and south, r being half the distance between the two positions of its suspending thread.

For repeated estimations it is most convenient to employ two magnetometers at once, a being taken as the mean of the two deflections. To eliminate dissymmetry of magnetism the two magnetometers are once made to change places. If the mean deflection in the normal position is a, and in the changed one a', the deflections in the first position must in future be multiplied by

$$1 + \frac{a' - a}{2a}$$

Corrections.—By the reaction of the needle on the magnet, and by the oblique position of the latter during simultaneous observation of a and ϕ, a slight correction is involved. If κ denotes the ratio of the magnetism of the needle (or of both needles together where two are employed) to that of the earth, the above value of T^2 must be multiplied by

$$\left(1 - 2\frac{\kappa}{r^3}\right)(\cos a - 2 \tan a \tan \phi).$$

As a rule this correction is small.

Scale-Distances.—If the scale-distances of the bifilar and unifilar are nearly equal, it is only necessary to measure the difference between them, which is easily done by means of stretched threads.

First Position.—The unifilar magnetometer may also be placed east and west of the bifilar magnet, in which case

$$T^2 = \frac{2D}{r^3(1+\Theta)(1-\frac{1}{2}\mathfrak{L}^2/r^2)} \frac{\tan a}{\tan \phi}$$

and the correction factor for T^2 becomes—

$$\left(1 + \frac{\kappa}{r^3}\right)(\cos a + \tfrac{1}{2} \tan a \tan \phi).$$

Compare F. Kohlrausch, *Wied. Ann.* xvii. 765, 1882.

Ring Magnet.—It has been proposed in this method to employ as bifilar magnet a wide ring of steel wire, which is suspended and magnetised horizontally, and can be inverted; and which deflects a small magnetometer in its centre. Adhering to the foregoing notation, in which r now represents the radius of the ring, if a and ϕ are nearly equal—

$$T^2 = \frac{D}{r^3(1+\Theta)} \frac{\tan a}{\sin \phi}$$

W. Stroud, *Proc. Roy. Soc.* xlviii. 260, 1890.

61.—Temporary Variations in Intensity of Terrestrial Magnetism. Durability of Magnets.

The magnetic variometer depends for its accuracy on the constancy of its magnetism, which is never complete. On a method of increasing it see **55a**.

I. *Bifilar Variometer* (Gauss).

A magnet is bifilarly suspended by two threads not widely separated. The lines joining the two upper and the two lower points of attachment of the threads are rotated with regard to each other till they form such an angle that the moment of rotation caused by terrestrial magnetism is balanced by that from the torsion of the threads and the weight of the suspended magnet when the latter is in an east and west position.

The slight rotation (read by mirror and scale) which is then caused by variations in the horizontal intensity of terrestrial magnetism may be taken as proportional to this variation. Increasing intensity moves the north pole of the magnet towards the north; it is therefore convenient when motion in this direction corresponds to increasing numbers of the scale.

Estimation of the Scale-Value E.—The change of intensity which corresponds to a deflection of the needle through one scale-division, expressed as a fraction of the intensity itself, is E. When therefore the scale division P corresponds to an intensity T, P' denotes an intensity T'.

$$T' = T[1 + E(P' - P)]$$

1. A magnet of polar separation $\mathfrak{L}$ ($\frac{5}{6}$ of its length) is allowed to act on the bifilar variometer from a sufficient distance r to the north or south, and in the same horizontal plane. Turning this magnet 180° (*i.e.* end for end) corresponds to a movement of the bifilar needle over n scale-division. If $\mathfrak{l}$ be the separation of poles of the bifilar needle the scale-value—

$$E = \frac{1}{n}\frac{4}{r^3}\frac{M}{T}\left(1 + \tfrac{1}{2}\frac{\mathfrak{L}^2}{r^2} - \tfrac{3}{4}\frac{\mathfrak{l}^2}{r^2}\right)$$

M is the magnetism of the deflecting bar, which need not be known absolutely, but only relatively to terrestrial magnetism, and may be determined by a simple deflection (**59**, II., **62**, II.)

Proof.—The bar M, acting from the great distance r, in its two positions increases or diminishes the intensity T by $2M/r^3$. Since the deflection is altered n scale-division by reversing the position of

M, one division denotes a change of $4M/nr^3$, or in parts of the intensity itself, $4M/(nr^3T)$, *Q.E.D.* On corrections compare **59**, II., and App. 15.

2. *With the Torsion Circle.*—If the instrument has a graduated torsion circle, E may be calculated from the angle of torsion a, that is, from the angle made by the vertical planes of the upper and lower points of suspension, as

$$E=\frac{1}{2A}\cot a,$$ where A represents the scale-distance.

The angle of torsion is measured by turning the magnet end for end (*i.e.* 180°) in the bifilar suspension, and rotating the torsion circle till the bar again lies east and west. The angle of this rotation is $2a$. The method assumes suspending threads of feeble torsional elasticity, *e.g.* long or thin brass wires.

The bifilar magnet stands always so nearly perpendicular to the meridian that the moment of rotation of terrestrial magnetism may be expressed by Tm. The bifilar moment of rotation is $D\sin a$ (**53**). Therefore $Tm=D\sin a$. If T now changes to $T(1+E)$ and a into $(a+1/2A)$; that is, when the instrument turns through one scale-division, we have

$$Tm(1+E)=D\sin\left(a+\frac{1}{2A}\right)=D\left(\sin a+\frac{1}{2A}\cos a\right)$$

Dividing both sides by $Tm=D\sin a$ gives E as above.

On the valuation of scale-divisions by torsion- and oscillation-observations, compare Gauss, *Result d. Magn. Vereins*, 1841, p. 1, or *Abh.* vol. v. p. 404, and Wild, *Carl. Repert.* xvi. 325, 1880, and further, F. Kohlrausch, *Wied. Ann.* xv. 536, 1882.

Temperature Correction.—A rise of temperature weakens the bar-magnetism, and causes the earth's magnetism to appear smaller. The expansion of the stirrup and wires has also a small influence. If μ be the temperature coefficient of the magnet (**62**A), β the coefficient of expansion of the stirrup, and β^1 that of the wires, 1° of temperature-change requires a correction of $(\mu+2\beta-\beta')/E$ scale-divisions. If stirrup and wires are of brass, the expression becomes $=(\mu+0{\cdot}000018)/E$.

II. *Deflection Variometer* (F. Kohlrausch).

A magnetic needle may be directed perpendicularly to the meridian by a deflecting magnet instead of bifilar suspension,

and similarly forms an intensity variometer. For temporary observation such an instrument is easily improvised.

Scale-Value may be determined exactly as in **1**, 1.

Four-bar variometer.—Four similar magnets are fixed on a horizontal frame capable of rotation, so that two are in the positions of the first, and two of the second arrangements. The former are at a distance 1·12 times greater than the latter, and therefore exert round the central point a directive force of the greatest possible constancy. This directive force must be somewhat greater than the earth's, which may be contrived by suitable placing of the magnets. In the centre, as magnetometer, is suspended a magnetic mirror, which is observed with a telescope and scale.

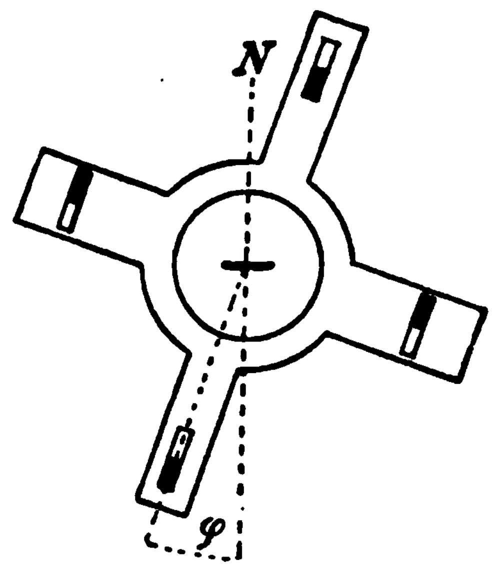


Fig. 42.

Adjustment.—The instrument is set approximately in the right azimuth, and its axis of rotation set vertical by means of the level and foot-screws. The exact adjustment in meridian is effected as follows: The frame is set to the zero of its divided circle, and in such a way that its magnetism works against that of the earth. The entire instrument is then turned till the needle takes the same direction as the magnetic bars, and is there screwed fast. The zero of the scale is now in the meridian.

The frame alone is now turned to an angle ϕ, at which the needle hangs vertical to the meridian, and is fixed in this position. If the scale-distance is A scale-division, the scale value is

$$E = \frac{1}{2A} \tan \phi$$

E may be made as large as is desired by so placing the magnets that ϕ is very small. This is the case when their directive force little exceeds that of the earth.

Temperature Correction.—Higher temperatures cause the earth's magnetism to appear too great. The influence is estimated in winter by alternate observations in cold and warm

rooms. If p_1 and p_2 are the scale-readings at temperatures t_1 and t_2, the correction for 1° is $(p_1 - p_2)/(t_1 - t_2)$. If at a future time another scale-value E' is employed, this expression must of course be multiplied by $\frac{E}{E'}$.

Compare F. Kohlrausch, *Wied. Ann.* xv. 540, 1882.

61A.—Comparison of the Horizontal Intensity in Two Places.

1. *By Oscillations.*

The same magnetic needle is allowed to oscillate at both places; the ratio of the intensities is as

$$T_1 : T_2 = t_1^2 : t_1^2$$

When accuracy is required, temperature and variations in terrestrial magnetism (**62A** and **61**) must be taken into account.

2. *By Deflections.*

If an identical east-and-west directive force deflects the needle a_1 and a_2 in the two places, then (**60**, 1)

$$T_1 : T_2 = tan\ a_2 : tan\ a_1.$$

Local Variometer (F. K.)—A much greater sensitiveness is attained when a magnetic needle is deflected about 90°, as in **61**, 2. The 4-bar variometer, as is obvious, may also be employed as a local variometer. We will here assume a simpler form with a rotatable magnet below a compass, but the first form, with mirror, is used in precisely the same way in principle.

1. *The Axis of Rotation* of the instrument is made vertical by the aid of a level.

2. *Suitable Distance of Magnet.*—The action of this when in the meridian must be somewhat stronger than that of the earth. For this purpose it is placed with its N pole to the north, and its distance from the needle is regulated till the north pole of the magnet points to the south. The greater

the sensitiveness required, the less must be the excess of directive force of the magnet over that of the earth.

3. *Adjustment to the Meridian.*—The magnet is turned to the zero of its divided circle, and the whole instrument is rotated till the needle stands parallel to the magnet. We will suppose that it is then also at the zero of the compass circle.

4. *Angle of Deflection ϕ of the Magnet.*—The magnet is turned to one side, till the needle points to 90°, and one stop of the magnet is fixed in this position. The adjustment is similarly made on the other side, and the instrument is ready for use. The half of the angle of rotation between the two stops is ϕ.

5. *Comparison of T in Two Places.*—The variometer is set up at place of comparison 1, adjusted as in 3 to the meridian, and the magnet is turned first to one stop and then to the other (4). We will read the needle-point at that side of the compass on which the numbers increase towards the north. Let the N pole of the needle read here p_n, and then, after turning the magnet, the S pole p_s, both being expressed in degrees of arc. Taking the difference, $p_n - p_s = \delta_1$.

Bringing now the variometer to point 2, we proceed exactly in the same manner, reading both poles in the different positions of the magnet, and taking the difference to which in this case we may assign the value δ_2. Then the relation of the two magnetic fields will be

$$\frac{T_1}{T_2} = \frac{1 + \tan\phi \times \tan\frac{1}{2}\delta_1}{1 + \tan\phi \times \tan\frac{1}{2}\delta_2}$$

for which, δ_1 and δ_2 are small, we may substitute

$$\frac{T_1 - T_2}{T} = [0{\cdot}0087 \times \tan\phi](\delta_1 - \delta_2)$$

The reduction factor $0{\cdot}0087 \times \tan\phi$ has the convenient value $0{\cdot}0050$ when $\phi = 29^\circ{\cdot}8$.

Proof.—Let the magnet exert on the position of the compass a directive force J on a unit needle. Then obviously $J \cos\phi = T$ where T denotes that strength of magnetic field in which the needle would be deflected 90° by the magnet turned to ϕ. Let then at place 1 a north-component $T_1 - T$ act on the needle, and perpen-

dicular to it a component $J \sin \phi = T \tan \phi$. If the needle then sets itself at an angle ϵ_1 with the east and west direction,

$$\tan \epsilon_1 = (T_1 - T)/(T \tan \phi), \quad \text{and } (T_1 - T)/T = \tan \phi \times \tan \epsilon_1$$

Similarly for the place 2. Since ϵ_1 and ϵ_2 denote our $\frac{1}{2}\delta_1$ and $\frac{1}{2}\delta_2$, the above equation is easily derived. The practical equation for small angles is obtained by setting $\tan \frac{1}{2}\delta = \frac{1}{2}\delta/57^\circ{\cdot}3 = 0{\cdot}0087\delta$ (p. 11).

Temperature.—The influence of temperature is determined in a warm and cold petroleum bath, in a manner similar to that described at the conclusion of **61**, and is taken into account in calculation. If the readings at the different places follow each other rapidly, it is better to keep the temperature of the magnets constant, wrapping them if necessary in felt or wadding.

Compare F. K., *Wied. Ann.* xix. 138, 1883; and xxix. 51, 1886.

62.—Determination of the Magnetism of a Bar in Absolute Measure.

I. The method described in article **59** or **60A** is completely applicable to this case. It is only necessary to eliminate T from the two numbers $MT = P$ and $\frac{M}{T} = Q$ by multiplication, and we obtain $M = \sqrt{PQ}$. But M, as we have seen, is the magnetic moment of the bar employed for deflection and oscillation, expressed in Gauss's absolute measure (compare App. No. 15, and Table 28).

The magnet employed on p. 245 has the magnetic moment

$$M = \sqrt{183{\cdot}01 \times 5388} = 993{\cdot}0 \ cm.^{\frac{5}{2}}\, g.^{\frac{1}{2}}\, sec.^{-1}$$

II. *Determination by Observations of Deflection.*

As the magnetism of bars varies with time and change of temperature, great exactness is seldom demanded, and since the horizontal intensity for the place of observation is approximately known (Table 22), the observations of deflection (**59**, II.) alone are often sufficient.

In most cases it is enough to observe a single deflection at one distance. If we call

r the distance of the centre of the magnet from that of a short needle;

ϕ the angle of deflection of the latter by the magnet;

$\mathfrak{L}$ the separation of poles—that is, $\frac{5}{6}$ the length of the magnet,

the magnetic moment M of the magnet is given by

$$M = \tfrac{1}{2} r^3\, T\left(1 - \tfrac{1}{2}\frac{\mathfrak{L}^2}{r^2}\right) \tan\phi$$

if the deflecting magnet be east or west of the needle, as in the figure on p. 241, or by

$$M = r^3\, T\left(1 + \tfrac{3}{8}\frac{\mathfrak{L}}{r^2}\right) \tan\phi$$

if it be north or south (p. 242). If a magnetometer be used, the expression must be multiplied by $(1 + \Theta)$ (**55**), on account of the torsion.

In the examination of a magnet not in the form of a bar, as, for instance, a magnetic mineral, the magnetic axis of which cannot be determined from its form, the body is turned into that position in which it produces the greatest deflection. By this means we obtain at the same time the position of the magnetic axis.

III. *Determination by Oscillation.*

In a bar of regular form we may calculate the moment of inertia K (**54**), and then we have from the time of oscillation t,

$$M = \frac{\pi^2 K}{t^2 T(1 + \Theta)}$$

IV. *Determination by Bifilar Suspension*

may be carried out according to **60A**, I.

V. *With the Balance* (Helmholtz).

Three bar magnets are required. Let the required magnetic moments be $M_1\, M_2\, M_3$, the polar separations being $\mathfrak{L}_1\, \mathfrak{L}_2\, \mathfrak{L}_3$

respectively. The bar M_1 is hung vertically from one end of a sensitive balance free from iron, and the bar M_2 horizontally from the other end, parallel to the beam, and at the height of the middle point of M_1. The balance is now brought into equilibrium. One of the bars is now turned so as to reverse the position of the poles, which destroys the equilibrium, and p grm. must be added to one side to restore it. Let the acceleration of gravitation be g (about 981 *cm./sec.*^{2}). The distance between the two knife-edges, which must be considerable in proportion to the length of the bars, is r cm. From this we may obtain the product of the magnetic moment of the two bars in absolute cm. g units by the formula—

$$M_1 M_2 = \frac{1}{12} \frac{r^4 p \cdot g}{1 - \frac{5}{2}\frac{\ell_1^2}{r^2} + \frac{10}{3}\frac{\ell_2^2}{r^2}} = P_{12}$$

To eliminate want of symmetry on the magnetisation, the experiment may be repeated, reversing the position of the other magnet, and taking the mean of the two values.

In a similar manner the values $M_1 M_3 = P_{13}$ and $M_2 M_3 = P_{23}$ may also be obtained.

From the three equations we have

$$M_1 = \sqrt{\frac{P_{12} \times P_{13}}{P_{23}}}, \text{ and so on.}$$

Compare Helmholtz, *Sitzungsber. d. Berliner Akad.* xvi. 405, 1883.

62a.—Temperature Coefficient of a Magnet.

The temperature coefficient may be defined as the decrease in the magnetism of a bar produced by a rise of 1°, divided by the total magnetism. The higher the specific magnetism, the smaller in general is the temperature coefficient. In good magnets it ranges from 0·0003 to 0·001.

The methods given in **62** will determine the relation of the magnetism of a bar to its temperature, but not with sufficient accuracy; and therefore we must increase the deflections produced by change of temperature.

I. *Compensation* (Weber).

The bar of which the changes are to be measured is placed so as to deflect a short-needled magnetometer from one side and at a moderate distance r, and the large deflection so produced is brought again nearly to zero by aid of a compensating magnet. The first bar is now raised to the different temperatures t_1 and t_2, and at each the deflection is read on the scale. Let n be the difference of the two readings, and A the distance of scale. The temperature coefficient μ will then be

$$\mu = C\frac{n}{t_1 - t_2}$$

the factor C being obtained as follows:—

1. If the magnet from a similar distance deflects a short needle ϕ, then

$$C = \frac{1}{2\mathrm{A}\ tan\ \phi}$$

2. If the magnetism of the bar is known, $\mathfrak{L}$ being the separation of its poles (p. 244), we have

for the first position $\quad C = \frac{T}{M}\frac{r^3}{4\mathrm{A}}\left(1 - \frac{1}{2}\frac{\mathfrak{L}^2}{r^2}\right)$

for the second position $\quad C = \frac{T}{M}\frac{r^3}{2\mathrm{A}}\left(1 + \frac{3}{8}\frac{\mathfrak{L}^2}{r^2}\right)$

3. The bar and the compensating magnet are alternately approached to the magnetometer in such a way that the approach of the one deflects towards one end of the scale, and that of the other in the opposite sense, till the needle is finally brought near to its original position. If N denotes the sum of the various changes of scale-reading caused by the approach of the bar (not of the compensating magnet), corrected according to **49**, p. 216, to quantities proportionate to the tangents of the deflections, then obviously

$$C = \frac{1}{N}$$

II. *By Bifilar Suspension* (Wild).

The bar to be examined is hung in a sensitive suspension directed from east to west, and the room is raised to different temperatures.

Let E be the scale-value determined after **61**, I.; then if a difference of temperature $t_1 - t_2$ makes a difference of position n,

$$\mu = \frac{nE}{t_1 - t_2} 2\beta + \beta'$$

β being the coefficient of expansion of the suspension, and β' that of the suspending wires.

The observations must only be undertaken at a time of great tranquillity of terrestrial magnetism, or its variations (**61**) must be taken into the calculation.

Compare Wild, *Carl. Rep.* ix. 277, 1873.

III. *By* 90° *Deflection of a Magnetometer* (F. Kohlrausch).

The bar is placed horizontally at the height of the (short) magnetometer needle, with its centre in the meridian of the needle, and so placed that, acting conjointly with the terrestrial magnetism, it makes it set east and west. The bar forms in this position the angle ϕ with the meridian. It is then warmed t degrees, which alters the reading of the magnetometer by the angle ϵ. The temperature coefficient is then

$$\mu = \tfrac{1}{2} \tan \phi \frac{\epsilon}{t}$$

For a small ϕ the method is very delicate.

On details and corrections compare F. Kohlrausch, *Wied. Ann.* xxii. 420, 1884.

62B.—Separation of Poles of a Magnet.

By poles are understood the points in which the two magnetisms of a slender bar may be conceived to be concentrated as regards their action at a distance, where the

fourth power of the relation of the length of the magnet to the distance may be neglected.

Let the magnet produce on a short needle at the same level, at the distances a_1 and a_2, measured from centre to centre, the deflections ϕ_1 and ϕ_2. Let the polar separation of the needle, that is, $\frac{5}{6}$ of its length, be $\mathfrak{l}$. Then calculate the expression—

$$\eta = a_1^2 a_2^2 \frac{a_1^3 \tan\phi_1 - a_2^3 \tan\phi_2}{a_2^5 \tan\phi_2 - a_1^5 \tan\phi_1}$$

The pole separation is then given, in accordance with Gauss's formula (p. 244), by the following expressions—

For observations in the first position $\qquad \mathfrak{L}^2 = +2\eta + \frac{2}{3}\mathfrak{l}^2,$
second position $\qquad \mathfrak{L}^2 = -\frac{8}{3}\eta + 4\,\mathfrak{l}^2.$

According to the altered formulæ (p. 244), we must reckon for the

first position $\qquad \mathfrak{L}^2 = 4\,\dfrac{a_1^{\frac{3}{2}} \tan\phi_1^{\frac{1}{2}} - a_2^{\frac{3}{2}} \tan\phi_2^{\frac{1}{2}}}{a_1^{-\frac{1}{2}} \tan\phi_1^{\frac{1}{2}} - a_2^{-\frac{1}{2}} \tan\phi_2^{\frac{1}{2}}} + \frac{3}{2}\,\mathfrak{l}^2$

second position $\qquad \mathfrak{L}^2 = 4\,\dfrac{a_1^2 \tan\phi_1^{\frac{2}{3}} - a_2^2 \tan\phi_2^{\frac{2}{3}}}{\tan\phi_2^{\frac{2}{3}} - \tan\phi_1^{\frac{2}{3}}} + 4\,\mathfrak{l}^2$

In order to eliminate unsymmetrical distribution of magnetism, the double deflection produced by turning the bar 180° (end for end) is always observed. It is also placed successively on both sides of the magnetometer, a_1 or a_2 denoting in each case half the distance between the two corresponding positions of the bar.

In exact measurements, the variations due to change of temperature of the magnet, and those of the earth's horizontal intensity, must be eliminated. This is accomplished most simply by the simultaneous employment of two magnetometers, on which the magnet acts from two positions symmetrical to the central point between them. If E be the distance of the two magnetometer threads from each other, and E' the distance over which the magnet is moved, $a_1 = \frac{1}{2}(E - E')$, and $a_2 = \frac{1}{2}(E + E')$. After the first set of observations, the magnetometers are exchanged, the observations are repeated, and the mean of the corresponding deflections is taken. The distance of the scale need only be known approximately. On reduction see **49.**

GALVANISM.

63.—General Remarks on Galvanic Work.

I. Ohm's Laws.

In Simple Undivided Circuits. Resistance, Current, Strength, Electromotive Force.

(1.) The electrical resistance w of a cylindrical conductor which is traversed uniformly by the current from end to end is directly proportional to its length l, and inversely to its sectional area q; or $w = \sigma \frac{l}{q}$. The factor σ varies in value in different substances, and is called the *specific resistance* of the body. As we ordinarily take $\frac{1}{w}$ as the *conductivity;* so we may call $\kappa = \frac{1}{\sigma}$ the *specific conductivity* of a conductor. The section of fluid columns is determined by weighing the contents of the tube (**19** and **19**A).

Resistance of Widenings.—If the current passes from the flat end of a cylinder of radius r into a wide space of specific resistance σ, the resistance will be the same as if the cylinder were lengthened by $0{\cdot}80 \times r\sigma'/\sigma$. If the space in which the current expands is filled with the same material as the cylinder, the equivalent lengthening amounts to $0{\cdot}80r$ (Rayleigh; compare also Maxwell, § 309).

Other Forms.—A conductor of any form, of which the places of ingress and egress of the current are exactly defined, has a definite resistance, which equals the specific resistance σ multiplied by a factor dependent on the form. This factor

is called the resistance-capacity of the space which the conductor occupies. In a cone, for example, of length l and radii of ends r_1 and r_2, when the current passes uniformly through the end surfaces, it $= l/(r_1 r_2 \pi)$; in a hollow cylinder of length h and radii r_1 and r_2, which is uniformly traversed radically by the current (as the fluid in a galvanic element of ordinary form), it is $= (\textit{log. nat.}\ r_2 - \textit{log. nat.}\ r_1)/(2\pi h)$.

Units of Resistance.—Those of practical importance are:

(*a*) The ohm (App. 21), which for the present is legally defined (in Germany) as a column of mercury of 1 sq. mm. in section, and 106·0 cm. in length at 0° C.

(*b*) Siemens's mercury unit, which is the resistance of a column of mercury of 1 sq. mm. section and 1 meter in length, at 0° C.

(*c*) The British Association unit = 1·0487 Siem. = 0·989 ohm. Thus we have

Legal Ohm : B. A. Unit : Siemens = 1·0600 : 1·0487 : 1.

Table 25 gives the specific resistance σ of the more important materials in terms of the ohm, as well as $\kappa = 1/\sigma$. The resistance of a cylinder of l meter length, and q mm.² section, is $\sigma \times l/q$, or $l/\kappa q$ ohm. By multiplication by 1·06 the resistance is reduced to Siemens's units. $\sigma/10{,}000$ is the resistance of a cube of 1 cm.³ in ohms, and is also called "specific resistance." $\sigma \times 100{,}000$ is the resistance of such a cube in absolute cm./sec. units, or the specific resistance in the cm. g. sec. system.

The last columns of Table 25, and for electrolytes, Table 26, contain the conductive capacity in relation to quicksilver at 0° C., $k = \kappa/1{\cdot}06$.

Resistance of a Copper Wire of 1 *m. Length and d mm. Diameter.*—Section $q = d^2\pi/4 = 0{\cdot}785 d^2$ mm.² Conductivity of the best conducting copper is $\kappa = 58$, or in terms of mercury $k = 58/1{\cdot}06 = 55$. The resistance is therefore

$$\frac{1}{58} \times \frac{1}{0{\cdot}785 d^2} = \frac{1}{46 d^2} \text{ ohm, or } \frac{1}{55} \times \frac{1}{0{\cdot}785 d^2} = \frac{1}{43 d^2} \text{ Siemens's units.}$$

If 1 m. copper wire weigh p grm., $d^2 = p/7$, and the resistance is $0{\cdot}15/p$ ohm.

1 $cm.^3$ of the best conducting sulphuric acid of 18^c C., of which $l = 0{\cdot}01$ meter, $q = 100$ $mm.^2$, $k = 0{\cdot}000069$ has the resistance

$$l/kq = 1/0{\cdot}69 = 1{\cdot}45 \text{ Siem. units} = 1{\cdot}45/1{\cdot}06 = 1{\cdot}37 \text{ ohm.}$$

(2.) The total resistance of a circuit is the sum of the resistances of all the separate parts.

(3.) The electromotive force of a battery is equal to the difference of potential or tension of its poles when the circuit is open and no current passing. The total electromotive force of a series of cells is the algebraic sum of all their separate electromotive forces. If the poles of a constant cell of electromotive force E and internal resistance w_0 be connected through an external resistance w_1, the potential of the poles $= Ew_1/(w_0 + w_1)$.

(4.) The current-strength or intensity i is directly proportional to the electromotive force E, and inversely so to the resistance w, or $i = C.E/w$. The numerical value of C depends on the units employed, which, according to Weber's system, are so chosen that unit electromotive force, acting against unit resistance, produces unit current. In this case $C = 1$ and $i = Ew$. Such systems of units are carried out in the "absolute" system of measures, as well as in the technical system, in which current-strengths are measured in amperes, resistances in ohms, and electromotive forces in volts (comp. App. 19-21). Thus

the current strength	1 am.	=	volt/ohm	$= 0{\cdot}1$	[cm. g. sec.]
„ resistance	1 ohm	=	volt/am.	$= 10^9$	„ „
„ electromotive force	1 volt	=	am. × ohm	$= 10^8$	„ „

These are the proper theoretical definitions. Since, however, the ohm is "legally" defined (in Germany) as $1{\cdot}06$ m./$mm.^2 Hg$ at 0° C. (probably about $0{\cdot}003$ mm. too small), it becomes necessary to distinguish between the value of the theoretical volt founded on the cm. g. sec. system, and the technical or legal volt. The technical volt hitherto adopted is on the above assumption about $\frac{3}{1000}$ smaller than the theoretical.*

* Probably the legal ohm will be altered to 1·063 m. Hg., and the ampere

The equation $i = E/w$ holds for a conductor of resistance w, even when it has in itself no *EMF*, if E denotes the difference of potential or tension between the two ends of w.

Hence follows the law, that in the Wheatstone bridge (Fig. at end of I.), when the bridge-current i vanishes, $w_1/w_4 = w_2/w_3$; since the potential at the two ends of the bridge-wire must now be equal, say P. If the potentials at the two other points of division are P_1 and P_2, then, since obviously $i_4 = i_1$ and $i_2 = i_3$, $(P_2 - P)/w_2 = (P - P_1)/w_3$, and $(P_2 - P)/w_4 = (P - P_1)/w_1$, from which the law follows.

Derived Currents in Divided Circuits.

If a current J between two points of the undivided conductor branches into several paths of the resistances w_1, w_2 .... and in which correspondingly we have the currents i_1 i_2 ......

(5.) The sum of the divided currents is equal to the undivided current, or $i_1 + i_2 \ldots\ldots = J$.

(6.) The divided currents are inversely proportional to the relative resistances of their respective paths (or directly to their conductivities), $i_1 : i_2 : \ldots = \frac{1}{w_1} : \frac{1}{w_2} : \ldots.$

(7.) The total conductivity of the divided circuit is the sum of all the conductivities of the single branches: $\frac{1}{w} = \frac{1}{w_1} + \frac{1}{w_2} + \ldots.$

Ohm's Law according to Kirchhoff.

The laws given above, from 2 to 7, are combined in the two following, which give directly the equations for currents in divided conductors.

A. At any point of division, the sum of the current-strengths in all the branches $= 0$, taking the currents *towards* the juncture as of opposite sign to those *from* it.

B. If we consider any part of the conductor which forms a

defined as that current which will separate 1·118 mg./sec. silver; but it will always be necessary to distinguish between theoretical and legal values.

closed circuit in itself, and reckon in it all the electromotive forces and currents in one direction as positive, and in the other negative, the sum of the products of the individual

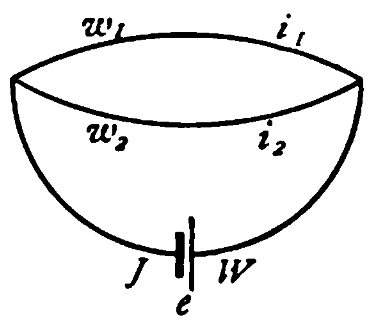


Fig. 43.

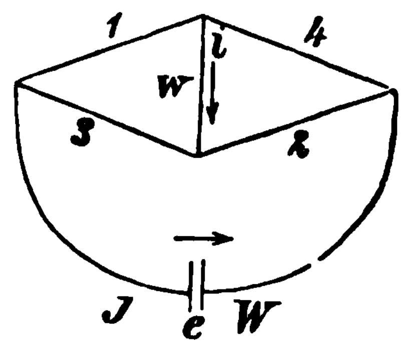


Fig. 44.

resistances into their respective current-strengths is equal to the sum of the electromotive forces.

For instance, in simple divided circuit, Fig. 43—

$$i_1 + i_2 = J, \qquad i_1 w_1 - i_2 w_2 = 0,$$
$$JW + i_1 w_1 = e,$$

whence

$$J = e \frac{w_1 + w_2}{Ww_1 + Ww_2 + w_1 w_2}$$

$$i_1 = e \frac{w_2}{Ww_1 + Ww_2 + w_1 w_2} \text{ etc.}$$

thus, for instance, $J : i_1 = (w_1 + w_2) : w_2$.

Or, again, in Wheatstone's bridge, Fig. 44, if we denote the divided currents and their corresponding resistances by similar numbers

$$\begin{array}{ll} J - i_1 - i_3 = 0 & JW + i_1 w_1 + i_4 w_4 = e \\ J - i_2 - i_4 = 0 & iw - i_1 w_1 + i_3 w_3 = 0 \\ i + i_1 - i_4 = 0 & iw - i_2 w_2 + i_4 w_4 = 0 \end{array}$$

Whence it follows that when the bridge-current $i = 0$, $w_1 w_2 = w_3 w_4$.

II. Galvanic Batteries.

As the fluid in which the zinc is immersed, dilute sulphuric acid is almost always used, seldom stronger than of a sp. gr. 1·06, *i.e.* about 1 volume of the strong acid to 20 volumes of water. For feeble currents a much weaker acid is mostly sufficient. The mixture of the acid and water produces a considerable rise of temperature, on which

account the acid is poured slowly with constant stirring into the water.

The solution of sulphate of copper in the Daniell's cell should be saturated (sp. gr. about 1·2, or 1 part crystallised salt to 3 parts water). It becomes exhausted by the current, and therefore the battery becomes inconstant. The strength of a Daniell's cell usually increases for a time on first setting up.

For bichromate cells, Bunsen prepares 1 lit. of fluid as follows:—92 grms. of powdered bichromate of potash are rubbed down to a uniform paste with 94 c.c. of strong sulphuric acid. To this is added 900 c.c. of water, keeping it stirred, and continuing the stirring until all is dissolved. If the zinc is to remain a long time in the fluid, this must be further diluted with water. Strong constant currents must not be expected from the bichromate battery. When the solution has become quite dark by use, the cells are enfeebled and inconstant.

For small currents of high *EMF*, the medical trough batteries of Spamer with chromic acid are convenient; and for electrometric charges the little Clark cells of Quincke.

Clark's Normal Cell.—Pure quicksilver is covered with a saturated solution of zinc sulphate made into a paste with pure Hg_2SO_4 (mercurous sulphate free from mercuric) and solid $ZnSO_4$. Pure zinc plunged into this paste forms the negative pole, while a platinum wire fused through the bottom of the cell, or carried down into the mercury in a glass tube from above, is the positive pole. The element is sealed with paraffin.

Helmholtz's Calomel Cell.—Zinc, 5 to 10 per cent solution of zinc chloride, finely powdered calomel, quicksilver. This gives feeble currents, which are very constant for a long period.

Zinc is amalgamated by first producing a clean metallic surface by brushing and dipping in dilute sulphuric acid, and then either rubbing on metallic mercury or dipping the zinc into a solution of mercuric chloride or nitrate. Zincs should be brushed and washed immediately after use.

Many carbons lose their efficiency by long use. We may try to clean them by filing off the surface or by heating.

Porous cells which are to be washed are best kept for a considerable time under water, after surface rinsing, and letting the water soak through. This prevents the efflorescence of the salts on the upper edges, which soon spoils the cell.

In putting up batteries, the porous cells must be moistened first with the dilute sulphuric acid, not with the copper solution or nitric acid. The cells should be filled $\frac{1}{10}$ to $\frac{1}{5}$ deeper with sulphuric acid than with the other heavier liquids, to lessen their diffusion to the zincs.

In order to cover platinum or silver foil with platinum black (to "platinise"), the foil is placed in a dilute solution of platinic chloride, to which a little hydrochloric acid has been added, and is then made the negative electrode of a current, or is touched under the surface of the fluid with zinc.

Electromotive Force of Cells.—The Clark-element, when rightly prepared, is said to be reliable to $\frac{1}{1000}$, but only for feeble currents (not exceeding at most 0·0001 amp.) At the temperature t, its $EMF = 1{\cdot}437 - 0{\cdot}0010\,(t - 15)$ legal volts (Lord Rayleigh; v. Ettinghausen). The *Physikalischtechnische Reichsanstalt* verifies Clark-elements.

Ordinary Daniell-elements have an *EMF* of 1·08 to 1·12 volts. Strong acids increase the *EMF*; strong copper solution may reduce it with weak currents. According to Kittler, pure amalgamated zinc, dilute sulpuric acid of 1·075 sp. gr. or 11 per cent H_2SO_4, and concentrated copper solution of 1·20 sp. gr., and pure copper deposited by the current, give an *EMF* of 1·18 volts, temperature having but little influence.

Bunsen's or Grove's cells give about 1·9 volts; fresh chromic acid cells 2·0 volts, and a well-charged accumulator nearly 2 volts, even with a strong current.

Within the customary dimensions, Daniell's cells have a resistance of 0·3 to 0·6 ohms, Bunsen's from 0·1 to 0·2 ohms. The character of the porous cell is of considerable influence.

Such cells as those of Smee, Leclanché, etc., are inconstant, that is, they give a higher *EMF* with open circuit or with feeble currents than with more powerful ones.

Small Electromotive Forces.—These may be obtained by

divided circuits. The circuit of a constant (Daniell) cell is closed through a resistance (rheostat or bright wire), and two points P_1 and P_2 of this circuit are employed as poles. If E be the EMF of the cell, R the total resistance of the circuit (rheostat and cell), and z the resistance between the points where the circuit is divided, the EMF between these points is given by the expression Ez/R, and the resistance by $z\,(1 - z/R)$.

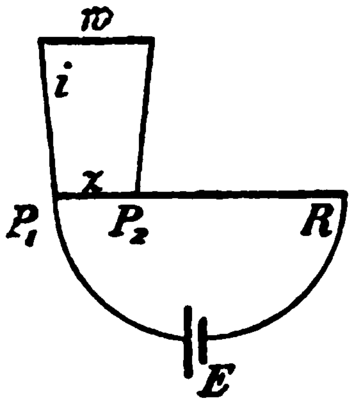


Fig. 45.

For if i be the current in an added circuit of resistance w (compare end of I.),

$$i = Ez : [w(R - z) + wz + (R - z)z] = Ez/R : [w + z(1 - z/R)].$$

Thermo-elements.—The thermo-electromotive force for 1° C. temperature is in absolute cm. g. units, or in 10^{-8} volts approximately for

Cu/Fe	*German sil./Fe*	*Pt/Fe*	*Pt/Pd*	*5 per cent IrPt/Pd*	*Bi/Sb*
1300	2500	1800	800	1200	9000

For *Cu/Fe* and *Pt/Fe* the EMF sinks with increasing temperature, for *Pt/Pd* it rises, for German silver/*Fe* it usually rises, but very slightly.

III. Galvanic Connections.

The simple touching of two solid conductors does not generally give a satisfactory connection. Where a firm connection cannot be made, the touching parts should be of platinum.

Even when using binding screws, the surfaces must be brightened, and the screws firmly tightened.

The plugs in rheostats are set in firmly with a slight turn in the hole. They should be frequently wiped with a clean cloth or with blotting-paper, and from time to time rubbed with the finest emery-paper.

Even mercury only gives a safe junction when the metal touching it (brass, copper, platinum, iron) is amalgamated. For this purpose the surfaces are cleaned with acid (platinum by ignition), and rubbing with mercury or dipping into a solution of that metal.

Platinum is best amalgamated by electrolysing an acid solu-

tion of mercurous nitrate. Iron must be previously tinned. Commutators require special care in this respect.

The contact of a metal with carbon should generally be over a large surface.

The Disturbing Influence of Conducting Wires on the needles of a galvanometer may mostly be avoided by placing wires carrying the current in opposite directions close to each other. In any case single wires should not be led near the needles, and large loops, especially vertical ones, should be avoided. The simplest commutator consists of a board with 4 mercury cups $\begin{smallmatrix}1 & 2\\ 3 & 4\end{smallmatrix}$, of which we can connect by means of a pair of metal bridges either 1 with 2, and 3 with 4, or 1 with 3, and 2 with 4. The wires from the battery are connected with 1 and 4, and the ends of the circuit with 2 and 3.

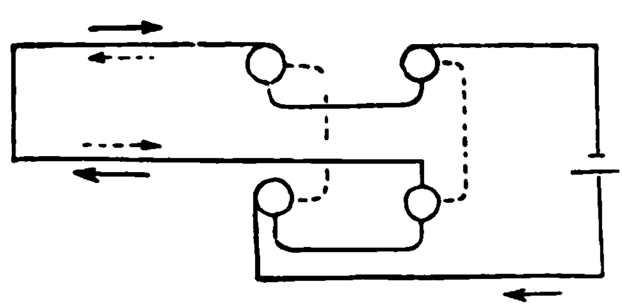


Fig. 45a.

IV. Galvanic Resistances.

The most important point in the choice of materials is their constancy, nextly, that they should be only slightly influenced by change of temperature, and finally, that they should have a high specific resistance. Ordinarily, alloys of copper, nickel, and zinc are employed, and more recently with the addition of manganese. German silver (Cu, Ni, Zn) has proved very satisfactory with regard to the constancy of its resistance. Its temperature-coefficient, *i.e.* the amount by which its resistance is increased for 1° C. expressed as a fraction of its whole resistance, is on the average 0·0004, but may vary between 0·00023 and 0·0006. In the alloys "nickelin" and "platinoid" it is about 0·00023. The smaller the conductivity, and the less is the influence of temperature (compare Table 25). Recently copper-nickel and copper-manganese alloys have been used to some extent, which in certain proportions (Table 25) have the great advantage of being scarcely influenced by temperature within wide limits. The constancy of the manganese alloy is, how-

ever, very uncertain (compare Feussner, *El. Techn. ZS.* 1892, p. 99).*

New wires suffer at first a noticeable change in resistance, and the winding itself is not without influence. Long-continued warming to 100° C. is said to favour the attainment of a constant state.

Resistance coils are wound "bifilar," that is, the wire is bent in its middle, and, beginning from that point, is wound double; or two wires are wound together and the ends soldered.

This arrangement has two advantages. The coils, when currents are passing, exert no magnetic influence around themselves, and, when the current-strength alters (as on closing and opening the circuit), they are not exposed to the disturbing electromotive forces of the extra current, which may easily lead to error.

On the other hand, long bifilar coils, especially if carefully wound, have a considerable electrical capacity, and the phenomena of discharge may similarly cause disturbances.

The resistance of a rheostat plug of the ordinary form, if carefully handled, should not exceed $\frac{1}{5000}$ ohm.

Small resistances are often conveniently arranged by connecting in parallel circuit. Small changes in a resistance, w, are most simply made by connecting a large resistance, R, in parallel, when the united resistance is $wR/(R+w)$, or approximately, $w(1-w/R)$.

Ten equal resistances, w, which can be arranged at will in parallel or series, give a choice of 94 different resistances between $10w$ and $w/10$ (compare F. K., *Wied. Ann.* xxxi. 600, 1887).

Heating by the Current.—In w ohms a current of i amperes evolves a heat of $0{\cdot}24\ wi^2$ gram-calories per sec. Wire of d mm. diameter without loss of heat would be warmed by i ampere per second; if copper, about $0{\cdot}008\ i^2/d^4$; if iron, $0{\cdot}06\ i^2/d^4$; and if good German silver, approximately, $0{\cdot}15\ i^2/d^4$ degrees. Resistance-wires for strong currents are supported in free air, or in an oil or petroleum bath. Net or

* "Manganin" resistances cannot be allowed to approach 100° C. without injuriously affecting their constancy at varying temperatures.—*Trans.*

perforated conductors are good from their rapid loss of heat. Freely stretched wires of copper or German silver, of 1 mm. diameter, change their resistance for a steady current of 1 am. about $\frac{1}{1000}$.

Shunts. Divided Circuits. — The frequently recurring necessity of dividing a current may generally be met by the use of a single rheostat (set of resistance coils), if it be led in at an appropriate point, and for this purpose suitable connections should be provided. It is very useful to have at least several plugs fitted for this purpose with connecting screws. The annexed figure shows how an ordinary rheostat can be so used to produce a shunt of (say) 10 ohms, with a parallel circuit of (say) 900 ohms. The arrows denote the principal current. The usefulness of a rheostat for such purposes is further increased when the single decades (tens, units, etc.) are divided by extra plugs, so that each division can be used independently. It is then possible, for instance, to insert a resistance in a circuit, to divert a branch circuit from a portion of the principal one, and in this also to insert a resistance (Hartmann and Braun).

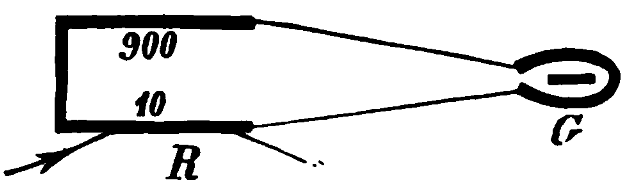


Fig. 46.

V. Efficiency of Batteries and Galvanometers.

Strong currents in conductors of low resistance depend principally on the size and nearness of the metal plates in the battery, and on the conductivity and degree of concentration of the copper solution or the nitric acid. For weaker currents in conductors of high resistance these circumstances are of less importance than the number of consecutive cells.

In a battery of several elements, when the greatest current in a given conductor is to be aimed at, they must be so arranged (by connecting the cells either consecutively or in multiple series) that the interior resistance may be as nearly as possible equal to the exterior. n cells have, when connected in single series, n^2 times the resistance which they have when all their similar poles are connected together.

The above rule for the maximum current assumes that the efficiency of the separate cells does not vary with the strength of the current. In reality, however, when the currents are strong we obtain better results when the interior resistance is smaller than the exterior.

The decomposition of water requires at least 2 Bunsen's or Grove's cells or 3 Daniell's.

Dynamo Currents are usually inconstant from the irregularity of the gas-engine or other motor. Increasing the moment of inertia by the addition of a fly-wheel is useful. The currents may be made very constant when accumulators in suitable number are employed parallel with the dynamo. For physical purposes, uniform tension machines (**77A**) are best suited. The dynamo should be so chosen that accumulators may be charged without direct coils, but with a purely "shunt" winding.

Accumulators.—The acid must always cover the plates by at least 1 cm. For refilling 5 per cent acid is generally suitable; but the strength of the added acid must be so chosen that the liquid in the charged accumulator has a density of 1·16, and in the uncharged of 1·13. The charging should always be continued if possible till gas is evolved. If the elements are not used, they should be charged weekly or fortnightly till gas is evolved, and should never stand uncharged. Cells which are short-circuited inside (which become warm in charging, and lose their charge rapidly) should be at once emptied.

The Thickness of Wire for winding galvanometers (or electro-magnets) should be so chosen that the resistance should be nearly equal to that of the remainder of the circuit. In a similar way the different coils often provided on galvanometers must be used parallel or in series, so as to attain the greatest sensitiveness.

The Magnetising Force (magnetic field), within a long bobbin, is at some distance from the ends, almost constant, and equals $4\pi ni$ where n is the number of windings per unit of length, and i the current-strength in absolute measure. In the end-surfaces the value is $2\pi ni$ (compare App. 19).

For methods of measurement in the forms which have become established for technical purposes, see, among others,

A. Kittler, *Hdb. d. Electrotechnik,* Stuttgart, 1886; Grawinkel and Strecker, *Hilfsbuch für die Elektrotechnik,* Berlin, 1888; Uppenborn, *Kalender für Elektrotechniker.*

64.—Tangent-Galvanometer (Pouillet and W. Weber).

The tangent-compass, or tangent-galvanometer, consists of a wide multiplier fixed with the plane of its coils in the magnetic meridian, and with a short needle in the centre.

I. *Relative Measurement of Current.*

For many purposes relative measurement, or determination of the ratio only of current-strengths is sufficient.

If two currents passed through the galvanometer deflect the needle respectively a and a', their relative strengths (intensities, quantities of electricity in unit of time) are proportional to the tangents of the angles of deflection, or—

$$i : i' = \tan a : \tan a'$$

Narrow coils of ellipsoidal form also cause deflections following the law of tangents (Riecke).

II. *Absolute Measurement of Current.*

The magnetic or Weber's unit of current-strength may be defined as that current which exerts a unit magnetic force. From a measurement by the tangent-compass the value of the current may be calculated in this unit in the following manner. Calling—

n the number of the circular windings of the multiplier;

R their mean radius in millimeters;

H the horizontal intensity of terrestrial magnetism (**59** and Table 22);

ϕ the angle of deflection of the needle;

then the strength i of the current which produces this deflection is, in magnetic measure—

$$i = \frac{RH}{2n\pi} \tan a = C \tan a$$

and we may call $\frac{RH}{2n\pi}$ the reduction-factor of magnetic measure.

Torsion of Thread.—If the needle be hung by a thread of torsion-coefficient Θ (**55**), $H(1+\Theta)$ must be written in place of H.

Proof.—The current i traverses the length $2nR\pi$ at the distance R from the short needle M, and tends to turn the needle perpendicularly to the plane of the coils, and exerts on it, when deflected through the angle α from their plane, a moment of rotation $iM \cdot 2nR\pi/R^2 \cos \alpha = iM \cdot 2n\pi/R \cos \alpha$. α is also the deflection from the magnetic meridian, and the earth's magnetic force produces a moment of rotation, $MH \sin \alpha$, in the contrary direction. The formula is obtained by combining these results (compare App. 16 and 19).

Different Current-units.—If R and H be measured in mm., mg., the results are given in the unit formerly used by Gauss and Weber. From the now customary measurement in cm. g. a unit 100 times greater is obtained, which is denoted by [cm.$^{\frac{1}{2}}$ mg.$^{\frac{1}{2}}$ sec.$^{-1}$], or, shortly, [cm. g.] (App. 19). That the reduction-factor for [cm. g.] is 100 times less than for [mm. mg.] is obvious, since R and H are each ten times smaller (**59**).

The current 1 ampere is $\frac{1}{10}$ part of 1 [cm. g.], therefore the reduction-factor for the tangent-galvanometer to amperes, when the measurements are in cm. g. is

$$C_A = 5\,\frac{RH}{n\pi}$$

Determination of the Radius.—This is either measured directly with rule, compasses, tape, or comparator, or calculated from the length l of the wire, which forms n coils as $R = l/2n\pi$. Thin wires must be measured and wound at the same tension.

The Intensity of Terrestrial Magnetism.—The reduction-factor of a tangent-compass will, of course, vary with time and place, since it is dependent on the intensity of terrestrial magnetism. For places where H has not been determined, it may be taken from Table 22—all local influences, such as iron objects, and especially long iron conductors, being as much as possible removed from the neighbourhood.

The place may be tested as regards constancy of H and compared in this respect with one out of doors by **61A**.

Example.—A multiplier is formed by winding a wire 1948·0 cm. long in 24 circular coils. Then $R = \frac{1948 \cdot 0}{48 \times 3 \cdot 1416} = 12 \cdot 92$ *cm.* Taking $H = 0 \cdot 1920$, the strength of a current which produces the deflection a is, in magnetic measure in mm. mg. system—

$$= \frac{12 \cdot 92 \times 0 \cdot 1920}{2 \times 24 \times 3 \cdot 1416} \tan a = 0 \cdot 01645 \tan a \text{ [cm. g.] or } 0 \cdot 1645 \tan a \text{ am.}$$

Advantageous Deflection.—An error of reading of 0°·1 causes (comp. p. 8)

for a deflection of	5	10	15	20	30	40°
or	85	80	75	70	60	50°
an error in result of	2	1	0·7	0·55	0·4	0·35 %

Hence both very small and very large deflections are disadvantageous to accuracy. For 30 cm. diameter and for currents of about $= i$ ampere, coils $n = 5/i$ are suitable. It is necessary, therefore, for currents of very different intensities, to employ galvanometers of different degrees of sensitiveness; that is, with coils of different diameters or of different lengths, or the instrument may be so constructed that the current may be passed through a greater or less number of coils as required, and preferably so that these coils can be used either parallel or in series. The results of different instruments may be compared with each other by passing the same current through both at once. If, for instance, we obtain in this manner a deflection of a_1 in I. instrument and of a_2 in II., the tangents of the angles of deflection of I. must be multiplied by $\tan a_2/\tan a_1$ to make the results comparable with those of II. Similarly, different coils of the same instrument are compared by passing the same current through them in series, in the same and in opposite directions.

Tangent galvanometers have been constructed with multipliers which can be inclined; in this case C must be increased proportionally to the reciprocal cosine of the angle of inclination (Obach).

Formulæ of Correction for Length of Needle and Section of Coils.—It is assumed in calculating the above formula that the section of the coil is very small compared with its diameter. It frequently happens that this condition is imperfectly ful-

filled in coils of many windings. If the coil is of rectangular section of breadth b and thickness h, a correction of the first order may be made by multiplying C by $1 + \frac{1}{8}\frac{b^2}{R^2} - \frac{1}{12}\frac{h^2}{R^2}$ (F. K.)

If the length of the needle is not very short compared to the diameter of the coil, we must add to the above expression the factor $\left(1 - \frac{3}{16}\frac{\mathfrak{l}^2}{R^2}\right)$, and, instead of *tan* a, must write $\left(1 + \frac{15}{16}\frac{\mathfrak{l}^2}{R^2} \sin^2 a\right) \tan a$. Here $\mathfrak{l}$ is the polar distance of the needle; which in slender needles is about $\frac{5}{6}$ of their actual length (**55A** and App. 15).

The complete formula, assuming that the corrections are small, will be—

$$i = \frac{RH}{2n\pi}\left(1 + \frac{1}{8}\frac{b^2}{R^2} - \frac{1}{12}\frac{h^2}{R^2} - \frac{3}{16}\frac{\mathfrak{l}^2}{R^2}\right)\left(1 + \frac{15}{16}\frac{\mathfrak{l}^2}{R^2}\sin^2 a\right)\tan a$$

Tangent Galvanometer with Rectangular Hoop.—R is the mean of the inner and outer radii, h the thickness.

1. Instead of $-\frac{1}{12}h^2/R^2$ substitute $-\frac{1}{6}h^2/R^2$ on account of the distribution of the current.

2. The hoop is divided, and has "leads" parallel to the middle radius, of the length l and their central lines separated a; in this case $+ al/2\pi R \,.\, (R + \frac{1}{2}l)/(R + l)^2$ must be added within the brackets of the correction. F. and W. Kohlrausch, *Wied. Ann.* xxvii. 21, 1886.

The corrections for length of needle disappear when $a = 27°$.

A needle of $l = \frac{1}{6}R$ gives deviations from the law of tangents up to 1 per cent. Excentric suspension of the needle about $\frac{1}{4}$ the diameter of the coils, to one side of their plane, greatly diminishes this error, but absolute measurements of current-strength cannot well be made (Gaugain, Helmholtz).

Leads.—The reduction formulæ assume that only the current in the coils of the instrument acts on the needle. Especially where the coils are few, care must be taken that the currents in the external conducting wires connected with the instrument do not affect the needle. This is most certainly accomplished by placing those leading to and from

the instrument close beside each other; or still better, twisted together.

Commutator.—If the plane of the coils is not accurately in the magnetic meridian, or the suspending thread is strongly twisted, deflections, especially if large, are too great to one side and too small to the other. By this the right position may often be more accurately fixed than by adjusting to zero, which in a short needle is unreliable. The correct deflection is obtained when the current is passed successively in opposite directions and the mean of the two deflections taken. It is convenient for this purpose that a commutator (**63**, III.) should be permanently connected with the galvanometer, which will allow the current to be reversed without altering any other part of the circuit. This gives the additional advantage of a double degree of accuracy, and renders it unnecessary to observe the zero-point exactly; and lastly, a well-arranged commutator serves conveniently to open and close the circuit.

Reading.—Two pointers perpendicular to the axis of the needle (glass threads) are convenient. For exact measurement the two opposite points should always be read. (Compare p. 151.) Parallax is avoided by laying a small piece of looking-glass on the galvanometer.

To bring the needle to rest, a small magnet may be employed, which is brought near or withdrawn as required. The commutator may also, with practice, be employed for the same purpose by reversal of the current, the circuit being at first merely interrupted, and only closed at the instant when the needle begins to return from an oscillation to the opposite side.

On the employment of mirror for reading see **48** and **49**.

III. *Measurement of Strong Currents by "Shunts."*

The following remarks apply not only to the tangent-galvanometer, but also to other types. If the galvanometer is too sensitive, it is possible for many purposes to remedy it, by passing a portion only of the current through the galvanometer, and the remainder through a suitable shunt or resistance

connecting the binding screws of the galvanometer. So long as the resistance of the shunt bears the same relation to that of the galvanometer (temperature !) the instrument may be used for *comparison* of currents just as without one.

For absolute measurement the relation of the resistance z of the shunt to that γ of the galvanometer must be known; the entire current is greater than that measured in the proportion

$$v = \frac{z+\gamma}{z} \text{ or } 1 + \frac{\gamma}{z}.$$

The current as calculated from the deflection of the galvanometer must therefore be multiplied by this shunt-factor v. If

$$\gamma : z = 9 : 1 \text{ or } 99 : 1, \text{ etc., } v = 10, 100, \text{ etc.}$$

Small resistances must be so intercalated in the circuit that the resistances of the connections are not injurious. This is the case, for instance, when the connection for a large weakening of current is made as shown in the figure. The entire current is passed through a short wire, while the connecting wires of the galvanometer are also inserted in the binding screws. In order that the required shunt resistance should not be too small, a resistance may also be added to the galvanometer, which is then included in γ (Fig. to **63**, IV.) The metal of the shunt wire must be insensitive to change of temperature, or so thick that it is not warmed by the current to a disturbing extent.

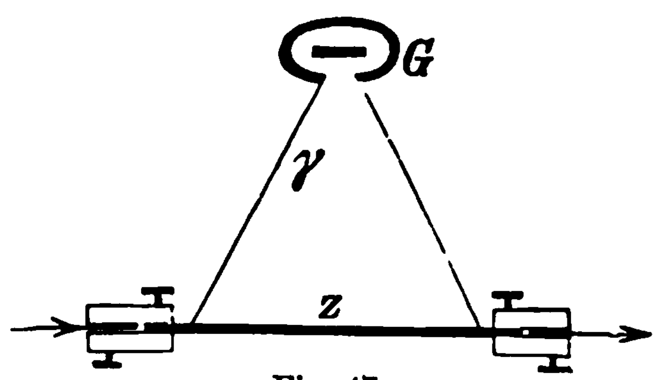


Fig. 47.

In order to measure currents up to 200 am. with a tangent-compass with a single coil, we may take n (say 10) wires of equal length, and solder $n-1$ (say 9) well together at the ends so that they lie parallel. The remaining wire, divided in two, and also soldered at the ends, joins the connection with the galvanometer, as much being cut away as corresponds to the resistance of the hoop of the instrument. The shunt-factor is then n (say $= 10$). Wires of 2 mm. diameter will suffice, since all are equally heated.

On the arrangement compare F. K., *Electrotechn. Zeits.* 1884, p. 13.

65.—Sine-Galvanometer (Pouillet).

In measurement, the relative position of the needle to the coils is kept constant by turning the latter after the deflected needle. The moment of rotation exerted by the current on the needle is therefore proportional to the current-strength. Since the opposing moment of rotation of the earth's magnetism is proportional to the sine of the angle of rotation a, we have

$$i = C \, . \, sin \ a$$

Since the sine at most $=1$, the limits of application are small. If the needle has also a divided circle, stronger currents can be measured with inclined positions of the needle (say 45° and 70°). In order to determine the reduction-factor between observations in two different positions of the needle, the angles of rotation of the coil a_1 and a_2 for the same current at different inclinations of the needle may be determined. Then the factor $P = sin\ a_1/sin\ a_2$.

On the absolute determination of c compare **69**. The advantage of the sine-galvanometer is the strict accuracy of the law of sines; disadvantages are the tedious adjustment and the double sources of error.

On the related torsion-galvanometer see **77**.

66.—Mirror-Galvanometers.

When the observations are limited to the small angles which are observed with mirror and scale (**48**, **49**), the current is proportional to the tangent of the angle of deflection a, or, for angles not exceeding a few degrees, practically to the deflection e measured in divisions of the scale, therefore

$$i = C \, . \, tan \ a, \text{ or also, } i = c/2Ae = C \, . \, e$$

if A be the distance of the scale. On the determination of the reduction-factor in absolute measure compare **69**.

The limits within which this proportionality may be assumed are, generally speaking, the wider, the shorter the needle, and the larger the coils. Close and broad coils are,

however, also favourable. The deviation from it is approximately proportional to the square of the deflection, or $i = Cn(1 + C'n^2)$. This may be tested, or the reduction-factor C' determined, by the use of a constant battery (Daniell), which is closed through the galvanometer and different rheostat resistances. The current is then inversely proportional to the total resistance (battery, galvanometer, and rheostat). For sensitive galvanometers, the required rheostat resistance is so large that a merely approximate knowledge of the two former is sufficient.

On a more exact simple method compare F. K., *Wied. Ann.* xxvi. 431, 1885.

Mirror-Galvanometers with Movable Coils (Wiedemann) are empirically adjusted. The deflections are compared for the same current with the coils in many positions of the graduated bar, and the results are tabulated graphically or otherwise. If r be the radius of the coils, and a its distance from the short needle, the deflections are approximately proportionate to $(a^2+r^2)^{-\frac{3}{2}}$.

On commutators compare **64**, II.; on the measurement of stronger currents by dividing the circuit, **64**, III.

66A.—Electrodynamometer (W. Weber).

The dynamometer consists of a fixed and a movable coil of wire, the latter being normally at right angles to the plane of the former, and both of which are traversed by the current to be measured. Directive force is given to the movable coil either by bifilar suspension, or by the elasticity of torsion of a suspending wire.

I. *Dynamometer with Deflection.*

The small deflections a of the movable coil (measured with mirror and scale) are proportional to the square of the current-strength i, or

$$i = C\sqrt{a}$$

where C is a factor for the particular instrument. The sensitiveness of the instrument is varied by altering the distance of

the bifilar suspension, or, in single wire instruments, by changing the suspending wire. C then increases with the square of the thickness of the wire, and is inversely proportional to the period of oscillation. On the absolute determination of C compare **69**.

Change of direction of the current in the entire instrument does not alter the direction of deflection. Only the outer coil is therefore connected with the commutator. For small currents the dynamometer is insensitive, since the deflection is proportional to the square of the current-strength.

For exact measurements precautions are necessary in respect of terrestrial magnetism and the elastic after-action (" memory ").

Alternating Currents.—The most frequent application of the instrument is for the measurement of currents which follow each other rapidly in alternate directions, but of equal strength. The dynamometer does not measure the strength of such currents in the ordinary sense, but the mean energy of the current. This is proportional to the sum of the products of the squares of the current-strengths and their related time-elements, spread over the unit of time, or equal to the mathematical expression $\frac{1}{t}\int_0^t i^2dt$, where i is the current-strength, and r its period. Indeed the square root of this expression is called the " mean current-strength " of an alternating current.

In alternating currents the self-induction of the coils must be taken into account. Especially the division of the current between the instrument and a shunt may vary much from that calculated from the resistances.

Further, it must be remembered that when the coils are not accurately perpendicular to each other, currents in the one exert induction on the other. To prove whether they are perpendicular, an alternating current is passed through the outer coil only, while the inner forms a closed circuit in itself, when the latter should not be deflected.

II. *Dynamometer with Zero-Reading* (Siemens).

The current-strength is measured by the angle of torsion

ϕ of an elastic suspension-spring, by turning the torsion-circle of which, the deflected coil is brought back to zero. The current-strength is $i = C\sqrt{\phi}$.

The axis of the movable coil must lie north and south, so that it may be unaffected by terrestrial magnetism. The mercury in the connecting cups must be clean.

On the determination or control of C compare **69.**

III. *Electrodynamic Balance.*

The force exerted by a fixed upon a movable coil is measured. The best form is that of Lord Rayleigh. A flat coil is hung from a balance between two larger equal and flat coils, of which the axes all lie in the same vertical. The separation is so regulated that the force is a maximum. The force acting on the movable coil is equal to the square of the current-strength multiplied by the number of windings and a factor, which essentially may be calculated from the relation between the radii of the two coils. The double force is measured, which is exerted on the balance when the direction of the current is reversed.

On absolute measurement with the balance compare Mascart, Exner's *Repertorium*, xix. 220, 1883; on the above arrangement of Lord Rayleigh's, *Phil. Trans.* ii. 411, 1884; especially also Heydweiller, *Wied. Ann.* xliv. 533, 1891, where an arrangement of a similar electrodynamometer without a balance is also described.

67.—BIFILAR-GALVANOMETER (Weber).

The current i passes through a coil hung by two conducting-wires, and of which the plane of the windings is north and south. The total area of the windings being f (**83**), fi is the magnetic moment of the coil, and terrestrial magnetism H causes the moment of rotation fiH.

D being the directive force of the bifilar suspension, determined after **53** from the weight and the measurement of the wires, or from the time of oscillation and the moment of inertia, an angle of deflection α corresponds to current-strength

$$i = D/fH \, . \tan \alpha.$$

Absolute Measurement with the Tangent Compass and Bifilar-Galvanometer.—Since H occurs in the numerator of the reduction-factor of the tangent-compass (**64**, II.), it is possible by the simultaneous employment of the two instruments to measure a current absolutely without knowledge of the terrestrial magnetism (compare **77**B).

Finally, the area of the windings of the bifilar galvanometer also falls out when the following method is employed. Let a tangent-compass of n coils of R diameter be placed at the distance a north or south of the bifilar galvanometer, so that its needle is deflected by the current in both instruments. Let Φ be the angle of deflection when both causes are acting in the same direction, and ϕ that when the current is passed through the tangent galvanometer only. We then obtain i independent of f and H by the following equation—

$$i^2 = \frac{R^2 D}{8\pi^2 n^2 a^3} \frac{(\tan\Phi - \tan\phi)^2}{\tan\Phi + \tan\phi} \tan a.$$

Proof simple.—On some corrections compare **77**B.

67A.—Other Forms of Current Measures.

(1.) *Galvanoscope with Close Coils.*—If this is used for large deflections it must be graduated empirically by comparison with one of the above instruments, or with the voltameter (**68**, **69**). The current-strength is generally no simple function of the deflection. Galvanoscopes with astatic pairs of needles usually suffer change of sensitiveness by passage of strong currents. Galvanometers with slow-swinging needles are called "ballistic."

(2.) *Needles with Horizontal Axes.*—These are under the directive influence of the current, of terrestrial magnetism, and of gravity. For constancy of indications it is necessary not only that the magnetism of the needle and the position of its centre of gravity in relation to the axis should remain unaltered, but in most cases that its position with regard to the magnetic meridian should not be changed.

(3.) *Current Measures with Directive Force by a Magnet.*

—Especially for the measurement of powerful currents, a stronger directive force than that of terrestrial magnetism is given by suitable steel magnets near the needle. The indications of such instruments are less influenced by disturbances of terrestrial magnetism, but are dependent on the constancy of their magnets (see **55a**.)

(4.) *Current Measurer with Soft Iron.*—Instruments in which the current acts on soft iron, first magnetising and then deflecting or attracting it, are unchangeable with time, and sufficiently accurate for many measurements. For feeble currents the forces are approximately proportional to the square of the current-strength, and in consequence the deflections are too small for use. Alternating currents act on such instruments, but it would be scarcely possible to graduate accurately for them.

(5.) *Galvanometer with Soft Iron Needle* (Bellati).—A suspended iron wire needle forms an angle of about 45° with the plane of the coils. A current magnetises the iron, and in consequence increases its deflection. The direction of deflection is independent of that of the current, and the instrument may therefore be used for alternating currents. An ordinary galvanometer with needle in oblique position is also affected by alternating currents (Cheesman).

Compare Giltay, *Wied. Ann.* xxv. 325, 1885.

(6.) *Spring Current-Balances.*—An iron body, suspended in a vertical coil by an elastic spring, is drawn in more or less deeply according to the strength of current. The elasticity of steel springs is very constant. Those of German silver used for smaller forces have a slight elastic after-action ("memory"). The graduation of the instrument is empirical (**69**). The indications are very constant if the iron is pushed deeper into the coil before reading, otherwise with increasing current it hangs slightly behind, and the indications are a little low. For weak currents the remarks above are applicable. Permanent steel needles are also adapted for use with weak currents. If long unused, they must be remagnetised by a strong current through the coil.

(7.) *Optical Telephone.*—A mirror is connected with the armature of a telephonic arrangement in which the length of the image of a small source of light can be observed. The period of oscillation of the armature can be varied, and the instrument reacts with alternating currents which synchronise with it.

M. Wien, *Wied. Ann.* xlii. 593; xliv. 681, 1891.

68.—Current Measurement with the Voltameter (Faraday).

If the products of chemical decomposition produced by a current be measured by a voltameter, they always bear an exactly defined relation to the current-strength, and form a measure comparable with the magnetic by the aid of the following rules:—

(1.) The decomposition in a given time by different currents is proportional to the current-strength.

(2.) The decomposition-products of the same current in different electrolytes are chemically equivalent (Faraday's law).

(3.) A current of 1 am. or 0·1 [$cm.^{\frac{1}{2}}g.^{\frac{1}{2}}sec.^{-1}$] decomposes or separates

	Silver.	Copper.	Water.	Mixed Gases at 0° C. and 760 mm.
in 1 second	1·118 mg.	0·3279 mg.	0·0932 mg.	0·1740 c.c.
in 1 minute	67·1 „	19·67 „	5·59 „	10·44 „

This quantity, the electrochemical equivalent (Weber) of a substance for 1 am., may be called A.

The current i which is to be measured is passed through the fluid for the time τ; the decomposed or separated quantity is m. Then the current-strength is

$$i=\frac{1}{A}\frac{m}{\tau}\text{ am. or}=\frac{1}{10A}\frac{m}{\tau}\,[\text{cm.}^{\frac{1}{2}}\text{g.}^{\frac{1}{2}}/\text{sec.}]$$

I. *Silver Voltameter.*

15 to 30 per cent solution of silver nitrate, of sp. gr. 1·15 to 1·35, with a silver anode. The precipitate on the kathode is weighed. A convenient form is a silver or platinum crucible as kathode (Poggendorff); a rod of pure silver forms the anode. A suspended glass cup catches any portions which fall from the anode. The precipitate is washed with hot distilled water, till the cooled washings show no reaction with hydrochloric acid, dried by heat, and weighed about ten minutes after cooling. Strong currents are apt to cause the growth of silver threads on the kathode, which grow towards the anode, and vitiate the results.

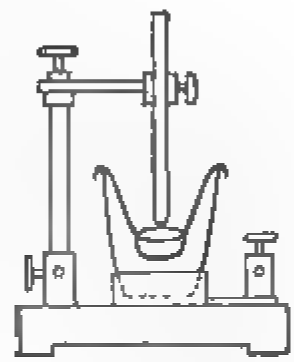


Fig. 48.

II. *Copper Voltameter.*

Almost saturated solution of pure copper sulphate in water; about 1 grm. cryst. salt dissolved in 3 c.c. water; sp. gr. about 1·16. Anode of pure copper; kathode, copper, or platinum. The kathode is rinsed and rapidly dried, first between blotting-paper, and then if possible under the air-pump, or in the desiccator, and weighed to determine increase.

The current-strength must be proportioned to the size of the electrodes; and in order that the precipitate should adhere firmly, must not exceed 1 am. for each 25 sq. cm. of the kathode. With weak currents too large electrodes may also cause losses proportional to the time, since the usually acid copper solution may dissolve a quite material weight of the electrodes.

Compare Gray, *Phil. Mag.* **xxv.** 179, 1888; Vanni, *Wied. Ann.* xliv. 214, 1891.

III. *Water Voltameter.*

10 to 20 per cent solution of pure sulphuric acid (1·07 to 1·14 sp. gr.), decomposed between bright platinum electrodes.

With strong currents the mixed gases are measured. With

closely approached electrodes of about 15 sq. cm. each, currents up to 40 am. may be measured without inconvenient heating. The instrument shown in the figure (from which the little stopper must be removed during use) is refilled by simply inverting.

The volume, measured by temperature t, under pressure of p_0 mm. mercury of 0° C., is reduced to 0° C. and 760 mm. by the formula (Table 7)—

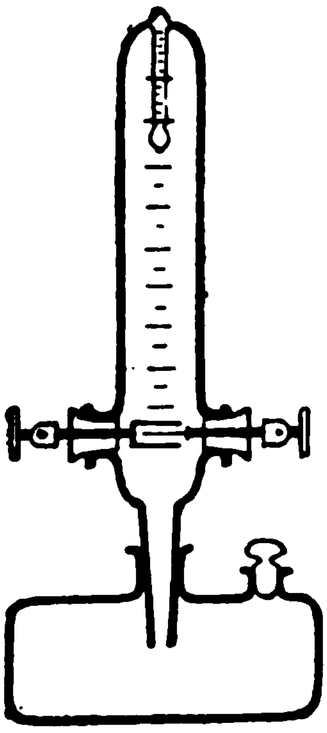


Fig. 49.

$$m = \frac{v}{1+0{\cdot}003674\,t} \cdot \frac{p_0}{760}$$

To find the pressure p_0, we take the height h of the acid in the tube above the free surface, S the density of the fluid, and b the height of the barometer (**20**); then from b must be deducted $Sh/13{\cdot}6$.

Secondly, we must deduct the tension of aqueous vapour in the gas. If the gas were collected over water, it would be saturated with aqueous vapour. The vapour tension over the acid is smaller, and its ratio k to the maximum tension e of the vapour (Table 13) at the given temperature is a proper fraction, and is for—

	0,	18,	27,	33 per cent H_2SO_4
or sp. gr.	1·0,	1·13,	1·20,	1·25
k	1·0 =	0·9,	0·8,	0·7

The pressure of the dry gases is therefore $p_0 = b - Sh/13{\cdot}6 - ke$, or very approximately $p_0 = b - h/12 - 0{\cdot}9\,e$.

Table for 15 *to* 20 *per cent Sulphuric Acid.*—The volume of moist mixed gases which are evolved by 1 am. current is not far from $\frac{1}{5}$ c.c./sec. under the usual conditions. The following table gives, for various pressures p, and temperatures t, the corrections which must be applied to the measured volume, in order to obtain the corrected volume v_0, with which to calculate $i = 5{\cdot}0\,v_0/\tau$ am.

For this purpose we must add to or subtract from each measured c.c. as many thousandths as are given in the table under the corresponding pressure p and temperature t, p

being the observed total pressure of the volume of gas, that is, $p = b - h/12$ (see above).

t	$p = 700$	710	720	730	740	750	760 mm.
10°	+ 9	+ 24	+ 38	+ 53	+ 68	+ 82	+ 97
15°	− 13	+ 2	+ 16	+ 30	+ 44	+ 59	+ 73
20°	− 35	− 21	− 7	+ 7	+ 21	+ 35	+ 49
25°	− 58	− 45	− 31	− 17	− 4	+ 10	+ 24

Compare F. K., *Electrotechn. Zeits.* 1885, p. 190.

With weak currents only the liberated hydrogen must be collected, and calculated to the volume of mixed gases by multiplication by $\frac{3}{2}$, because the oxygen is partially absorbed by the water, on account of the formation of ozone. The graduated limb of voltameter in the figure may be refilled with the liquid by merely turning on one side.

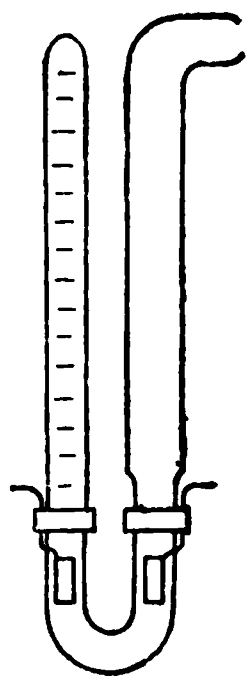


Fig. 50.

Since the polarisation of oxygen and hydrogen on platinum amounts to nearly 3 volts, at least three Daniell or two Bunsen cells are necessary for decomposition.

Example.—$v = 198$ c.c. mixed gases in $\tau = 117$ sec. at $t = 17°{\cdot}8$, and $b = 754$ mm. The column of liquid (20 per cent H_2SO_4) below the gas, $h = 112$ mm. The pressure of the damp gases, therefore, $p = 754 - 112/12 = 745$ mm. The tension of aqueous vapour at $17°{\cdot}8$ (Table 13), $e = 15{\cdot}1$, and the pressure of the dry mixed gases $p_0 = 754 - 0{\cdot}88 \times 15{\cdot}1 = 732$ mm. The volume of dry mixed gases is, therefore,

$$m = \frac{198}{1 + 0{\cdot}00367 \times 17{\cdot}8} \frac{732}{760} = 179{\cdot}0 \text{ c.c.}$$

and

$$i = \frac{1}{0{\cdot}1740} \frac{179{\cdot}0}{117} = 8{\cdot}79 \text{ am.} = 0{\cdot}879 \text{ cm.g.}$$

By the table the correction for $p = 745$ mm. is $= + 51$ at

15°, and $= +28$ at 20° C., and therefore at $17°\cdot8 = +38$. Therefore

$$v_0 = 198 \times 1\cdot038 = 205\cdot5, \text{ and } i = 5 \times 205\cdot5/117 = 8\cdot78 \text{ am.}$$

68A.—Current-Measurement with the Rheostat by a known Electromotive Force.

The current i, which is to be measured, flows through a resistance R, which can be varied at will. The two ends of R are also connected by a derived circuit which includes a galvanometer, and a known electromotive force E, which acts in the reverse direction to that of the current. R is varied till no current passes through G, when

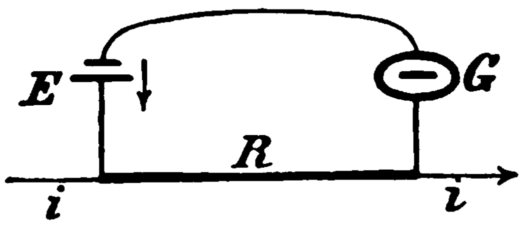


Fig. 51.

$$i = \frac{E}{R}.$$

From Kirchhoff's second rule, p. 267 B, it follows directly that in the circuit RGE, when there is no current in GE, $Ri = E$.

E and R measured in volts and ohms give i in amperes.

A Daniell or chromic acid cell, an accumulator, or, most reliable of all, a Clark cell may be used as E. On the electromotive forces of these elements see p. 269.

R may be an ordinary set of resistance coils, or a stretched bright wire with a movable contact, of which the resistance per unit of length is known.

Sources of error may arise, firstly, from the heating of R; and, secondly, the element E may be traversed by currents during the determination of the magnitude of R, which, if they are not very feeble, may weaken the electromotive force of a Clark cell. It is therefore advisable, during the rough testing of R, to insert a large resistance in the circuit of E (see figure and remarks, pp. 273, 274), which is removed in making the final adjustment.

On a compensation apparatus see Feussner, *Zeits. für Inst.* 1890, 1.

69.—Comparison and Absolute Determination of Galvanometer Constants, and Graduation of Current-Measures.

Empirical Determination of a Reduction-factor.

The law for the deflection of a galvanometer or dynamometer, etc., being known, the current-strength i is

$$i = C\,a \text{ or } C \tan a, \text{ or } C \sin a, \text{ or } C\sqrt{a}, \text{ etc.}$$

but in many cases the reduction-factor C cannot be calculated, and must, therefore, be determined empirically in one of the following ways:—

I. *With a Normal Galvanometer.*

These methods are also applicable to the comparison of the constants of two galvanometers.

(*a*) *In a Single Circuit.*—The instrument of unknown reduction-factor C is inserted in series in the same circuit with one of known factor C_1. If the latter indicates the current-strength i, and the former the deflection a, then according to the nature of the instrument,

$$C = \frac{i}{a} \text{ or } \frac{i}{\tan a} \text{ or } \frac{i}{\sqrt{a}}, \text{ etc.}$$

Or again, if a and a_1 are the deflections of the two instruments,

$$C : C_1 = a_1 : a$$

Instead of the deflections, or scale-readings, or angles of torsion, or weights, a and a_1, tangents, sines, square roots, etc., must be written according to circumstances.

(*b*) *By Successive Intercalation.*—Only accurate and convenient for very sensitive galvanometers. The same constant cell has its circuit closed first through one, and then through the other instrument. The two resistances being W and W_1, we have $C : C_1 = a_1 W_1 : a W$.

(*c*) *With Shunt.*—For instruments of very unequal sensitiveness. The instruments are connected in series, the more

sensitive with a shunt-circuit of resistance z (p. 281), while its own resistance is γ. Then

$$C : C_1 = a_1 : a \frac{\gamma + z}{z}$$

(*d*) *In Parallel Circuit.*—A current is divided through the two galvanometers, if necessary, with the addition of rheostat resistances. The total resistances of the two branches are w and w_1. Then

$$C : C_1 = a_1 w_1 : a w$$

The use of a commutator is mostly advisable, and especially to eliminate the action of the galvanometers on each other.

II. *With the Voltameter.*

A current is passed through the galvanometer and a voltameter for a measured time, the deflection a being noted, and the current-strength i through the voltameter being determined by **68**. The reduction-factor is then, according to the nature of the galvanometer,

$$C = i/\tan a, \text{ or } C = i/a, \text{ etc.}$$

Since a current, especially when passing through a voltameter, is seldom quite constant, the needle, etc., is observed from minute to minute, and the mean of the a, $\tan a$, etc., is taken (see above, on parallel circuits). If terrestrial magnetism comes into account, a commutator eliminates the effects of changes of the zero point.

III. *With a known Electromotive Force.*

(1.) *Direct.*—For sensitive galvanometers we may employ the very simple, and often sufficiently exact method of connecting up a battery of known electromotive force (Daniell, accumulators, or for the most sensitive instruments, Clark) through the galvanometer and a large known resistance. If the force be E volts, and the resistance W ohms, the current-strength

$$i = \frac{E}{W} \text{ am., and again, } C = \frac{i}{a}, \text{ etc.}$$

W is the added resistance of galvanometer + rheostat + battery. With very sensitive galvanometers the latter may often be neglected.

(2.) *With Shunt.*—It is shown in **68A** how a current-strength may be measured by a known electromotive force. It is only necessary to include the galvanometer to be tested in the circuit i with a suitable battery, and, if necessary, a rheostat for regulating the current. With reliable elements, and carefully employed, the method is very useful.

Compare among others, Grotrian, *Wied. Ann.* xxxi. 624, 1887. On testing the constancy of a reduction-factor by thermo-elements, see W. Kohlrausch, *El. tech. Zeits.* 1886, p. 273.

Graduation of a Current-Measurer.—This problem occurs with regard to instruments of which the action of the coils cannot be calculated, as is most generally the case. For the current-measurers mentioned in **67A** the entire scale must be graduated empirically. For this purpose the instrument is put in a circuit with a normal galvanometer as in I., or with a normal element as in III., and a number of readings are made with different current-strengths, perhaps with a temporary graduation in degrees or mm., and from these the divisions for round numbers of current-strength are interpolated. The most serviceable way of doing this is to set out on curve-paper the current-strengths as abscissæ, and the observations as ordinates, and through the points so found to draw a curve from which the required scale can be taken.

On the ballistic galvanometer, see **78A**.

70.—Determination of Galvanic Resistance by Substitution.

The proof of equality between like resistances is required both for copying resistances, and for the determination of an unknown resistance by comparison with one which we can increase at will by known amounts (set of resistance-coils, rheostat). We will first consider this simplest case.

Two resistances must be equal, which, when substituted for

each other in the same circuit, give the same current-strength. A circuit is formed, consisting of a galvanic cell E, a galvanometer G, and the rheostat R. The resistance, W, to be measured is shown in the figure as intercalated, but may be excluded by the use of a shunt without sensible resistance or other similar device. First, the position of the galvanometer-needle must be noted when W is included in the circuit and the rheostat plugged (cut out). W is then excluded, and an amount of rheostat resistance added in its place sufficient to bring back the needle to its original position.

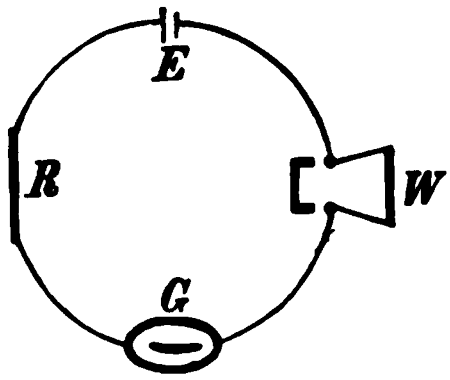


Fig. 52.

This added rheostat resistance is equal to the required resistance W.

If the resistance of the rheostat cannot be altered by sufficiently small intervals, but can only be altered by jumps—as in resistance-coils with plugs—we must make use of interpolation (5). The position of the needle is observed with the nearest resistances above and below that required, and if the difference of deflection is small, the increase of resistance may be taken as proportional to the decrease of deflection. If the observed position of the needle be

a with the required resistance W;
a_1 and a_2 ,, ,, rheostat resistances R_1 and R_2;

then

$$W = R_1 + (R_2 - R_1)\frac{a - a_1}{a_2 - a_1}$$

For accuracy and quickness, interpolation is always to be preferred.

Example—

Included in the circuit	W	R_1 14	R_2 15 ohms
Deflection	45°·3	47°·9	44°·5

$$W = 14 + \frac{2{\cdot}6}{3{\cdot}4} = 14{\cdot}76 \text{ ohms}$$

The method gives results of fair accuracy if the resistances are not too small. A *constant* element is necessary (Daniell).

Slight changes in the latter are eliminated by repeating the observation and taking the mean, and the disturbing effect is also diminished by rapid observations. For this reason it is best to make a rough measurement of W before the final determination.

When the resistance to be measured is small it is often necessary to include in the circuit a constant resistance in addition, because otherwise the galvanometer-needle would be deflected beyond the graduation. In this case, however, the measurement will be less sensitive. It is better, therefore, to bring back the needle to the graduation by means of a magnet placed near it, or, what is better, a smaller measured electromotive force is arranged as in Fig. 45, p. 271.

Shunt.—Lastly, it is often advantageous, especially with small resistances, to place the rheostat and the resistance to be measured in a shunt-circuit joining the two terminals of the galvanoscope, as shown in the figure. The equality of deflection shows that of the exchanged resistances as above.

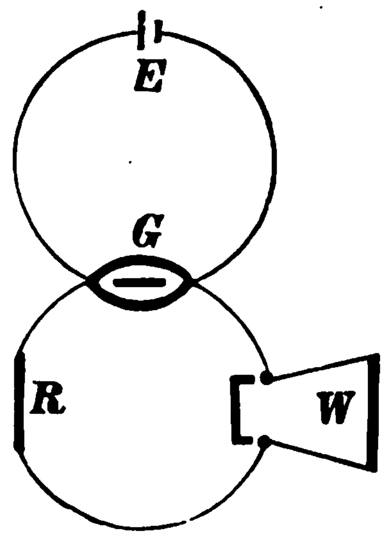


Fig. 53.

Specific Resistance.—If a wire of length l and sectional area q mm.^{2} has the resistance W, its specific resistance is $s = W \cdot q/l$ as compared with mercury if the rheostat used is based on Siemens's mercurial units. If the rheostat is in ohms, $\sigma = W \cdot q/l$ is the specific resistance in relation to the ohm which is, of course, 1·06 times smaller than the mercurial. $K = 1/s$ or $\kappa = 1/\sigma$ is the specific conductivity (comp. p. 265).

71.—Determination of Resistance by Measuring Currents.

Such methods are of importance, among other cases, in measuring the resistance of conductors which are influenced by the current, as, for instance, that of electric lamps while in use.

I. *Direct Method* (Ohm).

A circuit is made, including a battery and galvanometer,

and when necessary also an additional resistance, and the current-strength is measured. Let it be J. The resistance to be measured, W, is next included, and the current-strength again measured is i_0. A known resistance R is substituted for W, and gives a current-strength of i. Then

$$W = R\frac{J - i_0}{J - i}\frac{i}{i_0}$$

For J, i, i_0, we naturally take the angles of deflection, or their tangents or sines respectively. The method only gives good results with large resistances, as the electromotive force of almost all elements varies with the current-strength. If the resistances to be compared are very large, the remaining resistance may sometimes be neglected; in this case J disappears, and we have simply $W = R\ i/i_0$.

II. *Methods with Divided Circuits.*

1. A constant current is passed through a galvanometer G (tangent-compass) and the resistance to be measured, arranged in series. From the two ends of the resistance, a derived circuit is carried through a second sensitive galvanometer g (mirror galvanometer), of which the reduction-factor in comparison with the principal galvanometer is known, and through a large added rheostat-resistance. Let γ be the total resistance of the derived circuit, J the strength of the principal current, and i that in the derived circuit. The required resistance is then

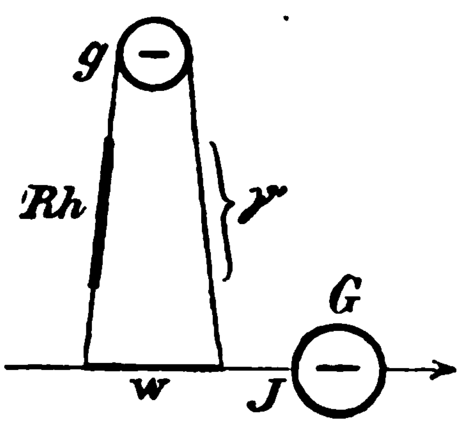


Fig. 54.

$$W = \gamma i/(J - i).$$

The comparison of the two galvanometers is inconvenient, but for many purposes, as, for instance, that of measuring the resistance of glow lamps, which require a powerful current, it is advantageous that only a small portion need pass through the comparison-resistance.

2. The derived circuit as above contains a potential

measurer ("voltmeter") (**76A**, **77**) of the resistance of γ ohms, and shows the potential P volts. The strength of the main current is J am. Then the required resistance is $\frac{P}{J-P/\gamma}$ ohm. This method is distinguished from No. 1 only by the name of "voltmeter."

3. The following method is of frequent and satisfactory application. The resistances to be compared, say a known and an unknown, are intercalated one behind the other in the same constant circuit. The two ends, first of the one, and then of the other resistance, are connected by a derived circuit of very great resistance, and containing a sensitive galvanometer or voltmeter. Assuming that the resistances to be compared are very small in relation to that of the derived circuit, they bear the same proportion to each other as the observed currents or potentials in the latter.

The employment of a commutator to the galvanometer (or indeed to the entire derived circuit) is convenient, but the latter is only permissible when it is certain that the needle of the galvanometer is not influenced by the current in the principal circuit.

See also methods with the electrometer, **84** and **86A**.

The above equations are proved by the Ohm-Kirchhoff laws, **63**, I.

71A.—Differential Galvanometer.

I. *Comparison of Resistances.*

Two resistances are equal when, if inserted as two branches of a circuit, the current divides itself equally between them. The equality of the two currents is determined by the differential galvanometer (Becquerel), the coil of which consists of two wires of equal length wound together. One current passes through one wire, and the other through the other, in opposite directions; and thus, when the currents are equal, they neutralise each other's influence on the needle inside the coils. The two currents, therefore, are known to be equal when the needle is undeflected.

The annexed figure shows the connections for measurement

of resistances. *G* represents the two coils of the differential galvanometer, with their ends brought out (which, of course, may be differently arranged, and must be found by experiment).

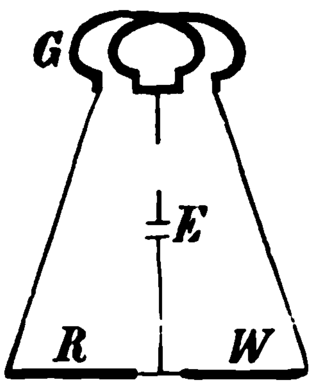


Fig. 55.

The current of the battery *E* divides between the two middle terminals, and passes through the coils in opposite directions. From the outer ends one-half of the divided current is led through *W*, the resistance to be measured, and the other through the rheostat *R*, uniting again at the opposite pole of the battery. The amount of rheostat resistance intercalated to bring the needle back to its normal position is equal to the resistance *W*. The method of interpolation may be employed here (p. 296). The connecting wires of *W* and *R* must be of equal resistance.

Testing the Differential Galvanometer (Bosscha).—In this method the differential galvanometer is assumed to possess two properties—first, that the current-strengths are equal when the needle is uninfluenced. This is tested by passing the same current through both coils in opposite directions; that is, counting the terminals of the galvanometer from left to right, 1 and 2 must be connected with each other, and 3 and 4 each with a pole of the battery. The needle should remain undeflected. Secondly, that the resistances of the two coils are equal. The previous requirement being fulfilled, this may be tested by allowing the current from one battery to divide itself through the coils, as in the figure just given, but without any resistance introduced when the needle must again be undeflected. Any required correction of the instrument should be made in the above order.

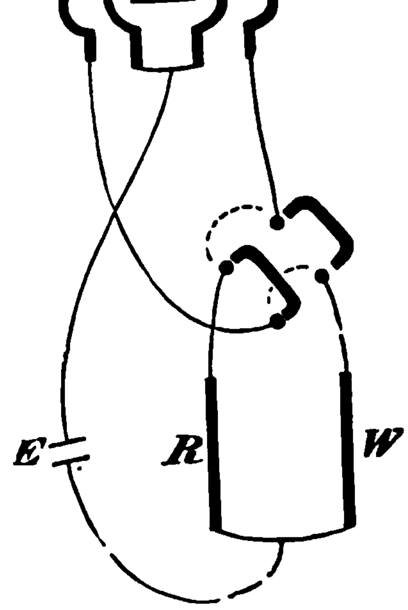


Fig. 56.

Commutator.—Lastly, we may be independent of the exact fulfilment of these conditions if we connect *W* and *R* with a commutator, so that their positions may be easily reversed. *W and R are equal when reversal of their position does not influence the deflection of the needle.*

Again, if *R* be a rheostat, and the needle remains at rest

when R_1 is inserted in one position of the commutator and with R_2 in the other, then

$$W = \tfrac{1}{2}(R_1 + R_2)$$

The advantages of the method are its sensitiveness and independence of the element.

Shunt Arrangement of Differential Galvanometer.—When the resistance to be measured is smaller than the resistance of one branch of the galvanometer a greater sensitiveness is attained by the following arrangement. The two branches of the galvanometer are included in the circuit of a battery, not side by side, but one after the other, but so that the current may traverse them in opposite directions. The two resistances W and R to be compared are arranged as "shunts" of the two galvanometer branches, as in Fig. 58 (Heaviside).

Small resistances frequently cannot be sufficiently securely connected. The shunt method renders connecting resistances to a great extent harmless, if the differential galvanometer has itself a great resistance (Kirchhoff).

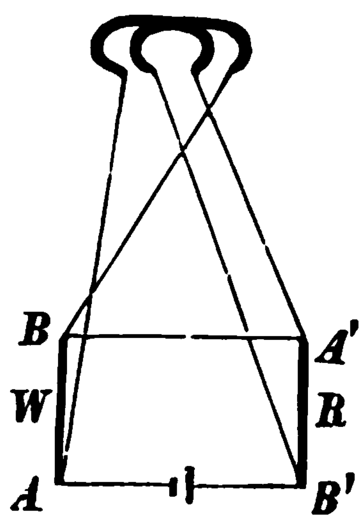


Fig. 57.

Crossed Shunts.—Resistances of connections are completely eliminated if the derived circuits in the previous figure are so changed that each multiplier is connected with both resistances (F. K.). Let it be found that no deflection takes place when W and R_1 are inserted. Now, by means of a suitable commutator, the connections of the source of current E with A and B' are reversed, without any other change in the circuit. The deflection is now zero when W and R_2 are inserted. Then—

$$W = \tfrac{1}{2}(R_1 + R_2)$$

Small errors of the differential galvanometer are also eliminated, so that resistances of 0·01 ohm can easily be compared to $\frac{1}{10000}$ of their value.

Compare F. K., *Wied. Ann.* xx. 76, 1883; *Exner Repert.* xix. 594, 1883.

Production of Small Changes of Resistance.—It is often not possible by ordinary methods to make slight changes in small

resistances, if we have none sufficiently small at our disposal. A small resistance, w, is, however, easily altered slightly by using a rheostat as a shunt. If the latter has the resistance R, the two together are $Rw/(R+w)$. In order to equalise two small and slightly unequal resistances, w and w^1, the larger, w, has R added as a shunt, so that the two together are equal to w^1. Then

$$w^1 = wR/(R+w).$$

II. *Comparison of Unequal Resistances* (Kirchhoff).

The two resistances W and R, which are to be compared, are connected, one behind the other, in a circuit, and from the ends of each of them a derived circuit is carried through the two coils of the differential galvanometer in opposite directions, sufficient resistance being inserted in the shunt of the greater resistance to cause the needle to show no deflection.

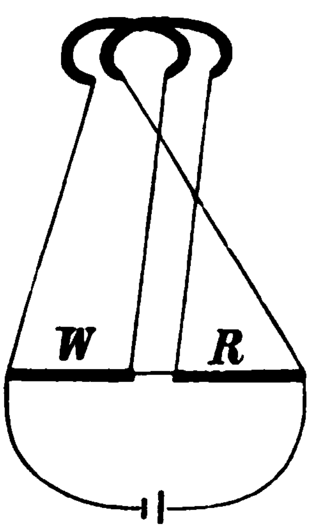


Fig. 58.

If now an addition of γ is made to the resistance of one shunt, ρ must be added to that of the other to bring the needle again to rest. Then

$$W : R = \gamma : \rho$$

for the currents in the derived circuits are equal when their resistances bear the same proportion as $W : R$. If these shunt resistances are at the first observation w and r, and at the second $w+\gamma$ and $r+\rho$, then

$$W : R = w : r = (w+\gamma) : (r+\rho) = \gamma : \rho.$$

The method eliminates the resistances of connections, but the differential galvanometer must be exactly adjusted to equal current-strength. Equal resistances are not required. By momentary closure of circuit extra currents may be produced, which disturb the results.

Compare Strecker, *Wied. Ann.* xxv. p. 464, 1885.

With the Differential Inductor.—Let an induction coil (**81**) consist of two equal wires wound with each other. Two opposite ends of the two wires are connected directly with

one terminal of a galvanometer, the other ends with the resistances to be compared, and thence to the other terminal of the galvanometer. If the two resistances are equal the needle experiences no current by an "induction impulse."

Wound resistances of many coils cannot be thus determined without further precautions on account of the extra currents.

71B.—Wheatstone's Bridge.

I. *Comparison of Resistance.*

With the current-division figured at the side, the current-strength in the branch G of the "bridge" is equal to zero if the resistances are in the proportion

$$a : b = c : d$$

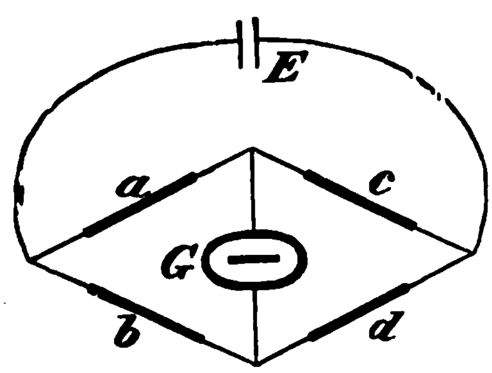


Fig. 59.

This follows immediately from the last equations on p. 268 by putting $i = o$.

If, therefore, a and b are two conductors of equal resistance, c the resistance to be determined, and d the rheostat resistance, and further, E represents a battery, and G a galvanometer, c will be equal to that rheostat resistance which must be introduced to make the current in G vanish. The arrangement of the resistances may also be so varied that the known equal resistances are in the branches a and c, while those to be compared are in b and d. If the resistance in the undivided conductor is greater than that in the bridge the arrangement $a = b$ gives the greatest sensitiveness and *vice versâ.*

The sensitiveness depends, of course, on the magnitude of the branch resistances as well as on their ratio to the resistances to be compared and to that of the galvanometer. It is well, therefore, to prepare separate pairs of equal resistances (*e.g.* 1, 10, 100, 1000 mercury units or ohms, of which the most suitable may be chosen for use.

Regarding the best arrangement of the measurements, cf. *Pogg. Ann.* vol. cxlii. p. 428, 1872.

Commutator.—We become independent of the equality of the resistances a and b by exchange. The resistances c and d are equal, if, when they are substituted for each other, the reading of the galvanometer is unchanged. Again, if d be a rheostat, and if before and after the substitution of c and d, the rheostat resistances required to bring the needle to rest are d_1 and d_2, then $c = \frac{1}{2}(d_1 + d_2)$. How a commutator is used for this substitution is shown by the figure.

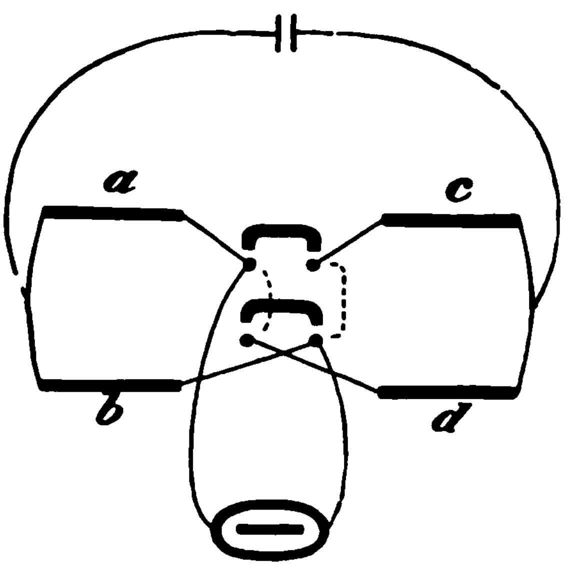


Fig. 60.

Interpolation.—If we cannot make up the exact resistance by the rheostat, we obtain it by interpolation from two approximating observations (5). With a commutator the following method is used:—The galvanometer readings e_1 and e_2 are observed with the nearly correct rheostat resistance R. This resistance R is then increased by the relatively small quantity δ and the readings $e_1{}^1$ and $e_2{}^1$ are observed. The figures 1 and 2 indicate the positions of the commutator. The desired resistance is then equal to

$$R + \delta \cdot \frac{e_1 - e_2}{(e_1 - e_2) - (e_1{}^1 - e_2{}^1)}.$$

Comparison after Foster.—In the figure, a and d are the resistances to be compared, and b and c two other nearly equal ones. AB is a wire stretched over a divided scale, on which a contact is movable, connecting it with the galvanoscope. Let the current in the bridge vanish when the contact is at x. Exchanging a and d, a new position x^1 of the contact is required to make the current cease. If r be the resistance of one scale-division of the measuring wire, and if the numbering increase from A to B, obviously $a - d = r(x^1 - x)$, for the entire resistance of the circuit $aABd$ is unchanged.

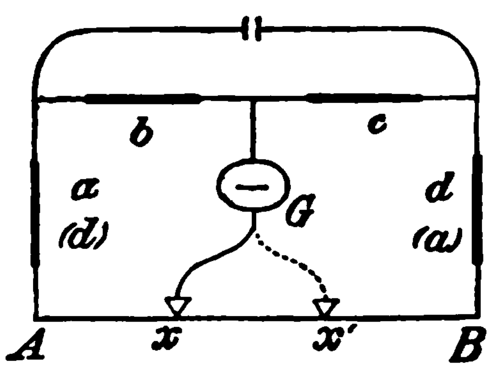


Fig. 61.

r is determined either after Matthiessen and Hockin,

p. 307, or a known resistance is measured as a, while a thick copper connection is used at d.

Momentary Closure.—In order to avoid the production of alterations of temperature by the current it is advisable, in using the differential galvanometer or the bridge, to use the current only for an instant, and therefore induction currents (**81**) may be used.

This method, however, causes errors if the resistance of a coil of wire is to be determined, because of the production of extra currents which influence the first deflection of the galvanometer. With Wheatstone's Bridge this source of error is avoided if we arrange, by means of a suitable key, that the connection with the bridge is completed an instant later than that with the battery. (In that in the figure, $a\,b$, $a\,b$ make the battery contact while a' and g complete that of the bridge. c is a piece of non-conducting material on which the finger is pressed.—*Tr.*)

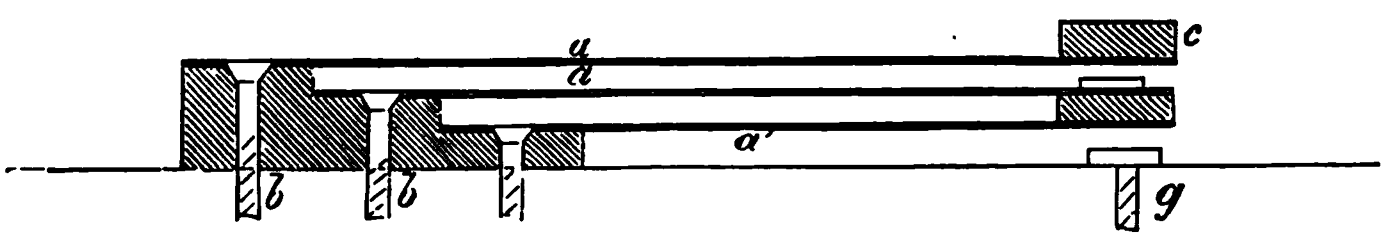


Fig. 62.

Measurement of very great or very small Resistances.—Here it may be advisable to choose the branch resistances a and b unequal in a known ratio (1 : 10, 1 : 100). In this case the possibility of a control by substitution is done away with.

When the resistances to be measured are those of short thick wires, it is often impossible to make their connections with the rest of the conductors sufficiently perfectly conducting. The following variation of the bridge connections makes us independent of such connection-resistances (W. Thomson).

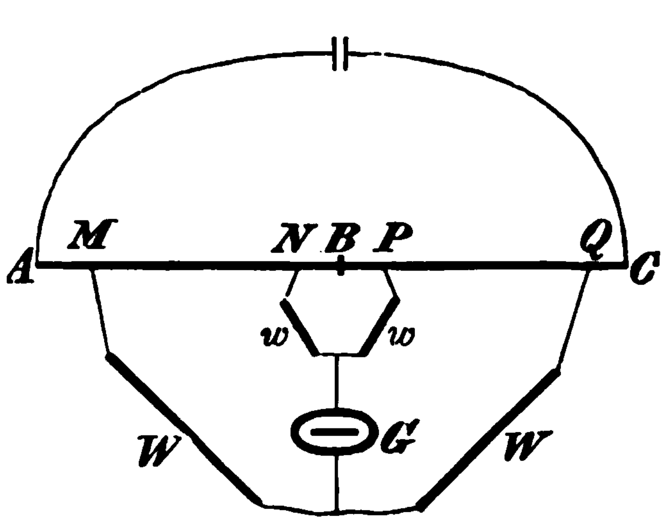


Fig. 63.

Let $A\,B$ and $B\,C$ be the wires to be compared, connected at B with each other, and at A and C with a cell. Further, let the two branches marked w and those marked W be respectively

equal resistances of not too small amounts. In G is a sensitive galvanometer. Four points M, N, P, Q are then found by trial, at which the last-named branches, well connected with the wires, make the current in G vanish. The resistances $M\,N$ and $P\,Q$ are then equal to each other.

II. *Comparison of Unequal Resistances* (Wheatstone-Kirchhoff).

In the figure a and b represent two resistances, of which the ratio can be easily varied. This is the case when a and b together consist of a stretched wire of uniform diameter of German silver, or still better, some alloy uninfluenced by temperature (Tab. 25), in which we may take the resistance as proportional to the length. On the wire is a movable contact (platinum), to which the connecting wire of the galvanoscope is attached. P_1 and P_2 are the source of current, and the galvanometer. It is indifferent in principle which of the two places is occupied by each, though under some circumstances the sensitiveness is greater in one or the other case. If the battery is at P_2, the sliding contact is more certain in action, which is a great advantage, while by the other arrangement errors due to heating of the wire are more easily avoided. The two resistances to be compared are inserted at W and R, and by experiment a position found for the contact in which no current passes through the galvanoscope.

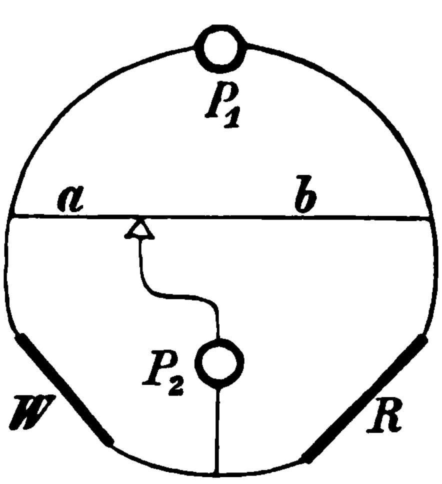


Fig. 64.

Then
$$W : R = a : b$$

On the calculation compare Tab. 37, and Obach, *Hilfstabeln*, Munich, 1879. The scale may also be divided to give the relation a/b directly.

The connecting wires of R and W have no influence when their resistances are in the same ratio as $R : W$. Hence it is advisable roughly to determine this ratio by a preliminary experiment, and to approximate to it that of the lengths of

wire (of the same sort) on each side. For this purpose it is convenient to join R to W by a single wire, and to connect the galvanometer wire to it by means of a movable binding screw.

The accuracy of placing the contact increases with the length of the wire. Stretched wires cannot exceed 1 m. long without inconvenience. The winding of the wire in a decimal number of turns on an insulating drum capable of rotation (wood, marble) allows the employment of much greater lengths, and is a great convenience. Contact is made by a spindle with a nut fixed on it which runs on the wire; and a connection with the axis without resistance by a large number of spring wires which press upon it. At R a plug rheostat can be intercalcated. W is the required resistance, and for the purpose of direct comparison another resistance can be inserted to the left. P_1 and P_2 are source of current and galvanoscope; compare on this point remarks above.

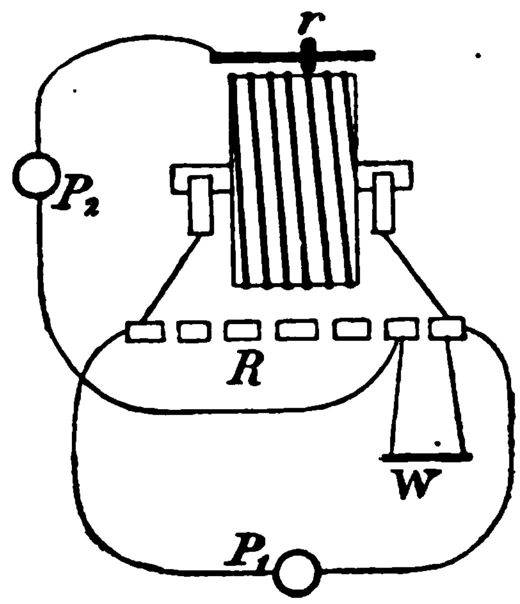


Fig. 65.

Additional Resistances.—Accuracy may be increased by resistances which can be inserted to right and left of the wire, of the resistance of which they are suitable multiples. In order to increase the accuracy of comparison in the neighbourhood of 1 : 1 and 1 : 10, for instance, two resistances, each of 4·5 times that of the wire, are needed, which are introduced, either on each side, or both at one side, as required.

See also the arrangement of bridge-wire of Siemens and Halske, etc.

Comparison of Nearly Equal Resistances.—The inequality of the two halves of the bridge-wire (which must always be expected) may be eliminated by reversing the position of the resistances, and taking the mean of a/b and b/a.

Calibration of Wire.—Compare **71D**, II.

Employment of Rheostat.—A rheostat may be employed in place of b and a known resistance (1, 10, 100) in that of a.

Comparison of Short Thick Wires (A. Matthiessen and Hockin).—Let A B and B C be the two pieces to be compared,

$D\ E$ a stretched bridge-wire. Taking a position P_1, a point M_1 is sought at which the current in the galvanometer disappears.

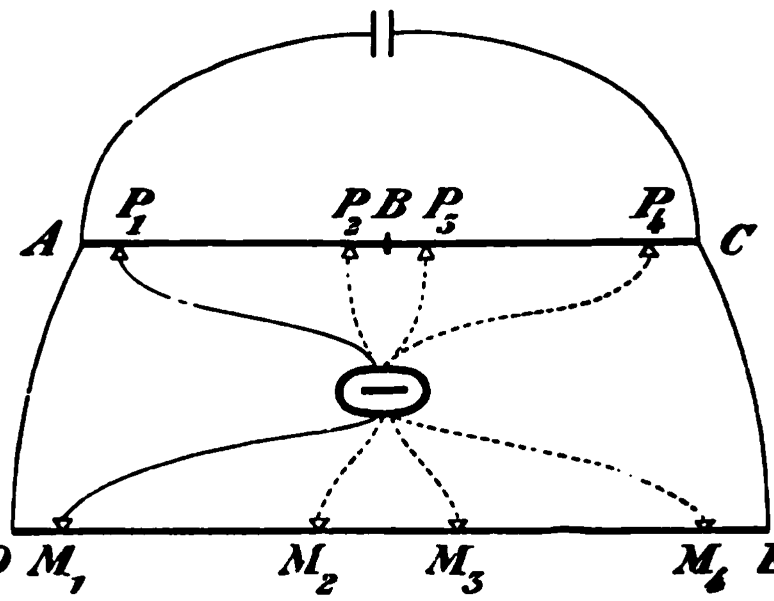


Fig. 66.

Similarly the pairs of points $P_2\ M_2$, $P_3\ M_3$, and $P_4\ M_4$ are determined. Then the resistances are in the ratio

$$P_1\ P_2 : P_3\ P_4 = M_1\ M_2 : M_3\ M_4,$$

for the vanishing of the current in G shows that in the corresponding points of contact there are equal electrical potentials (**63**, I. 4).

71c.—Comparison of Resistance by Damping.

A needle swinging inside a closed coil induces currents in it which react upon the needle in opposition to its motion. The logarithmic decrement (**51**) of small oscillations in a wide multiplier is constant and inversely proportional to the total resistance $\gamma + w$ of the coils and connecting wire (**78**, 7).

w_1 and w_2 are the resistances to be compared. The logarithmic decrement is observed:

λ_0 when the multiplier, of which the resistance is w_0, is closed by a wire of no sensible resistance;

λ_1 when the resistance w_1 is included;

λ_2 when w_2 is substituted for w_1;

λ' with the open multiplier, and through the mechanical resistance of the air;

then—

$$\frac{w_1}{w_2} = \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2} \cdot \frac{\lambda_2 - \lambda'}{\lambda_1 - \lambda'}$$

The above formula follows at once from the equation—

$$(\lambda_0 - \lambda') : (\lambda_1 - \lambda') : (\lambda_2 - \lambda') = 1/\gamma : 1/(\gamma + w_1) : 1/(\gamma + w_2)$$

Or if we wish to compare the resistance with that of the multiplier itself, by which it may be measured if the latter is known, or *vice versâ*, we have—

$$\gamma : w_1 = (\lambda_1 - \lambda') : (\lambda_0 - \lambda_1)$$

This method is only applicable to small resistances, since with large ones the damping is too feeble to be exactly measured.

The time of oscillation, and, at the same time, the damping, may be increased by astatising the needle (**55A**). When λ is large a correction of $\frac{1}{4}\lambda^3$ must be subtracted from λ.

71D.—Calibration of a Rheostat or Bridge-wire.

I. *Plug Rheostat.*

To test and form a table of corrections for a rheostat the parts of like nominal value are compared with each other. Simple substitution (**70**) gives results of no great accuracy.

In the employment of the differential galvanometer or bridge, the shunt-circuit passes out by one of the plug-blocks of the rheostat. If the rheostat has no provision for this (as should always be the case), a point must be sought at the attachment of the wire, or a place must be scraped bright. It is not necessary that this connection should be quite without resistance.

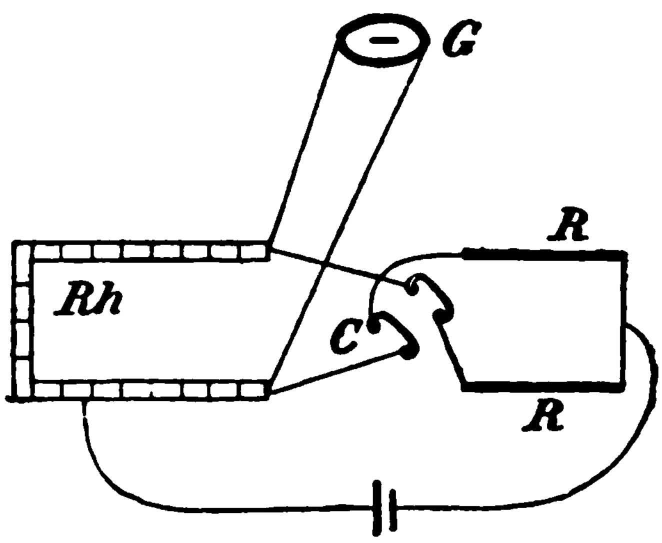


Fig. 67.

Comparison in the Bridge (**71B**, I.).—The pole-wires of the battery are connected between the two equal resistances R and R, and to a point in the rheostat, on both sides of which nominally equal resistances R_1 and R_2 are unplugged. In order to eliminate inequalities of R, R, the commutator *without resistance*, C is arranged to exchange the connections of R_1, R_2 with R, R. The positions of the needle e_1 and e_2 with the commutator in first and second position are observed. A

small known resistance δ (1, or 0·1, or 0·01) is added (if possible to the smaller resistance R_1), and again the needle-positions e'_1 and e'_2 are read. Then (p. 304)

$$R_2 = R_1 + \delta \frac{e_1 - e_2}{(e_1 - e_2) - (e'_1 - e'_2)}$$

Also a sufficiently long (p. 307) bridge-wire (lengthened at each end), in place of RR, gives very accurate results, and saves interpolation.

The Differential Galvanometer may be substituted for the bridge with about the same accuracy. For arrangement see **71a**, under "Commutator." RW is the rheostat.

Small Resistances.—For the sub-divisions 0·1 to 1·0 or 2·0 the differential galvanometer in shunt, or in crossed shunt (p. 301), is most accurate (compare same page on the production of small resistances); but the derived circuit method, **71**, II. 3, is simpler, and generally sufficient.

Calculation of Tables of Correction.—The customary order 5, 2, 2, 1 being assumed, the single resistances are marked and distinguished by indices. Taking a second 1 (for which the sum of the tenths may be used), we find by observation that

$$\begin{aligned} 5' &= 2' + 2'' + 1' + \alpha \\ 2'' &= 2' \qquad\qquad\quad + \beta \\ 2' &= 1' + 1'' \qquad\quad + \gamma \\ 1' &= 1'' \qquad\qquad\quad + \delta \end{aligned}$$

In addition it has been determined by comparison with a standard resistance, or with a higher section of the same rheostat, that the sum of the units has an error of ρ

$$5' + 2' + 2'' + 1' = 10 + \rho$$

The factor must now be calculated,

$$\sigma = \frac{\alpha + 2\beta + 4\gamma + 6\delta + \rho}{10}$$

from which is deduced the table of corrections (see **12**)

$$\begin{aligned} 5' &= 5 - 5\sigma + \alpha + \beta + 2\gamma + 3\delta \\ 2'' &= 2 - 2\sigma + \beta + \gamma + \delta \\ 2' &= 2 - 2\sigma + \gamma + \delta \\ 1' &= 1 - \sigma + \delta \\ \text{and } 1'' &= 1 - \sigma \end{aligned}$$

In the arrangement 4, 3, 2, 1 we compare 4 with $3+1$, 3 with $2+1$, 2 with $1+1'$, and 1 with $1'$, where $1'$ is the sum of the tenths.

The tens, hundreds, etc., are treated in the same way. In rheostats which are influenced by temperature it is advisable, for the sake of small corrections, to take the sum of the entire resistances (or the four largest) as correct. Normal temperature is then that at which this sum is actually correct. On temperature-coefficient see **71E** and Table 25.

The plugs must be carefully kept clean, and tightly put in. During rise of temperature they easily loosen themselves. A good plug has a resistance of $\frac{1}{5000}$ to $\frac{1}{10000}$ ohm.

On a source of error from unsuitable connections between the plug-blocks and the resistances see Dorn, *Wied. Ann.* xxii. 558, 1884.

II. *Calibration of a Wire.*

(1.) *With two Knife-Edges.*—A constant current is passed through the wire. A sensitive galvanometer of great resistance is connected with two knife-edges, which are fixed at a constant distance from each other. These edges are placed on different parts of the wire, and the indication of the galvanometer noted. These are proportional to the resistances of the portions of the bridge-wire between the edges (**71, 99**, II.) The constancy of the current must be tested; most simply, by returning from time to time to the same length of wire. In order to prove whether the wire is of uniform section, the knife-edges are moved along the wire, noting whether the galvanometer remains constant (Braun).

In roller wires, it is particularly easy to apply the above method to whole turns of the wire.

(2.) *With the Differential Galvanometer.*—The connections to each multiplier are provided with edges, which are placed on the wire so that the needle remains at rest. The two lengths have then the same resistance (p. 300). Very high resistance of the galvanometer is assumed, so that resistances of connections may be neglected. The following method is independent of these.

(3.) *By the Bridge Method of Matthiessen and Hockin*, p. 307

(Strouhal and Barus).—As many approximately equal resistances as the number of the sections of the wire to be tested are connected in series by mercury cups. The method is now carried out as shown by the Fig. 65, p. 307, the same one of these resistances being compared successively with each length of the wire, and moved forward a place after each determination. (*Wied. Ann.* x. 326, 1880.)

(4.) *After Foster.*—AB is the wire to be calibrated, $A'B'$ a second wire. P_1 and P_2 are the source of current and galvanoscope. The resistance W is a fraction, say $\frac{1}{10}$ or $\frac{1}{20}$ of AB; w is a stirrup of thick copper wire. W and w can be changed without resistance.

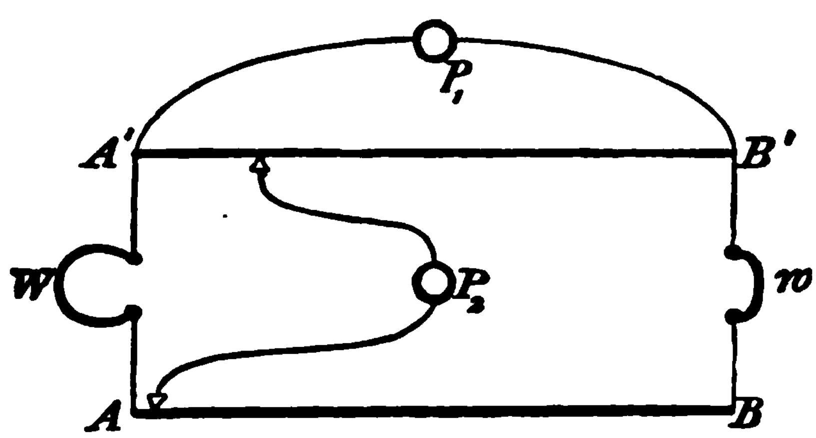


Fig. 68.

The lower contact (C) is first placed near to A, and the upper one (C') is moved till the current vanishes in G. W and w are now exchanged, C' allowed to stand, and C moved till the current vanishes; the resistance of the length of wire over which C has moved is obviously $= W - w$. C is now allowed to stand, W and w brought to their original positions, and C' moved till the current again disappears; W and w are again exchanged, and C moved over a second length of wire, which is again $= W - w$, and so on.

Foster, *Wied. Ann.* xxvi. 239, 1885; also earlier *Jour. Telegr. Eng.* 1872. See also, *loc. cit.*, the method of determining small resistances.

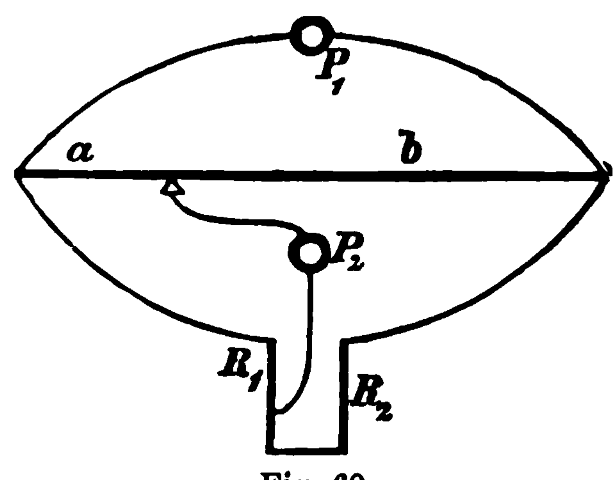


Fig. 69.

(5.) *With the Rheostat.*—If a calibrated rheostat is at hand, a table of corrections is simply made in the following manner. The rheostat resistances $R_1 : R_2$ are unplugged (perhaps successively $1 : 9$; $2 : 8$; $3 : 7$, etc., as a check, out of different decades), and the corresponding ratios $a : b = R_1 : R_2$ are determined. The

conducting wires to R and R are sufficiently thick for their resistance to be neglected, or they are determined and added. Points near the ends are determined with $R_1 : R_2 = 1 : 99$, and so on.

(6.) *With a Number of nearly Equal Resistances* (Heerwagen).—A number N (perhaps 10) nearly equal resistances (pieces of wire with amalgamated copper stirrups), connected in series with mercury-cups, give every ratio $m : n$ of the wire where $m + n = N$, in the following manner. Two groups of m and n resistances are compared with the wire. Single pieces are then exchanged between the two groups, and the comparison is again made, and so on till every piece has been included n times in the one, and m times in the other group. In this way we obtain $m + n$ independent positions of the contact, of which the mean divides the wire exactly in the proportions $m : n$.

Heerwagen, *Zeits. für Instr.* x. 170, 1889. Also discussion of the exactness of various methods, *ibid.*

71E.—TEMPERATURE-COEFFICIENT OF A CONDUCTOR.

The resistance of most metallic conductors increases with the temperature. If a conductor, at the temperatures t and t' has the resistances w and w', the temperature-coefficient is the factor a in the equation $w' = w[1 + a(t' - t)]$. If t t', w and w' are observed (**71A** and **B**), then

$$a = \frac{1}{w} \frac{w' - w}{t' - t}$$

For t a mean temperature of 15° or 18° is practically chosen.

For changes of temperature a petroleum bath surrounded with felt may be employed. If the coefficient is to be very accurately determined, a correspondingly delicate method must be chosen. With small resistances it is specially necessary to see to the constancy of the connections, and to exclude thermo-currents.

To examine a sample of wire, two equal pieces are cut off, of which one is placed in a constant cold bath, while the

other is heated and the resistances compared. The small differences are best measured by using a rheostat as a shunt to the greater of the two (compare end of **71A I.**).

If we have a normal wire of resistance W and temperature-coefficient A, the wire to be examined is brought to nearly the same resistance, and the two are warmed together. If at the temperatures t and t' the differences of resistance between the normal wire and that examined equal γ and γ', then

$$\alpha = A + \frac{1}{W}\frac{\gamma' - \gamma}{t' - t}$$

With great changes of temperature the coefficient is not quite constant. For exact representation we call w_0 the resistance at 0° C. and write

$$w_t = w_0(1 + \alpha t + \beta t^2 \ldots)$$

Measurement of very High or Low Temperature.—After determination of the resistance as a function of temperature, we may conversely employ the conductor as a measure of temperature. If the coefficients $\alpha, \beta \ldots$ are sufficient for a wide range of temperature, they are probably also approximately accurate for still wider limits. In this way, with, for instance, a platinum wire, temperatures may be measured which are beyond the range of any other thermometer. For this application a graphic representation of the temperature as a function of the resistance is most convenient, and from the curve a table may be constructed. (On the measurement of high temperatures compare also p. 98, and Callender, *Phil. Trans.* 1887, p. 161; and *ibid.* 1892, A, p. 119.)

Electrolytes.—The resistance of these diminishes rapidly with temperature. The temperature-coefficient in these cases is better referred to the change of conductivity, which generally appears more uniform than that of the resistance. If k and k' are the conductivities at t and t', we may write $k' = k[1 + \alpha (t' - t)]$, or more accurately, $k = k_0[1 + \alpha t + \beta t^2]$. On the measurement see **72**.

Temperature-coefficients of some substances are given in Tables 3A, 25, and 26.

71F.—MERCURY RESISTANCES (Siemens).

On the preparation of pure quicksilver see **19**.

Glass Tubes.—Tubes are mostly employed of sectional area between $\frac{1}{2}$ and 3 mm.^{2}, those of the most uniform bore being selected by means of a preliminary calibration with a thread of mercury. The ends of the tubes are ground flat, or slightly convex, on a disc of copper with fine emery on a lathe.

Measurement.—The length l of the tube may be measured to two glass plates attached to the ends with a very thin layer of cement. Two points opposite each other on the inner surfaces are measured (**18**). The mean area q is found from the weight of quicksilver filling the whole tube with plane ends (**19**).

The resistance-capacity of the tube is l/q if its form be strictly cylindrical.

Caliber Correction.—On account of the unequal section a caliber-factor C comes in, which is greater than 1. A thread of mercury, which on the average is n times shorter than the tube, takes in adjacent portions the lengths $\lambda_1, \lambda_2, \ldots \lambda_n$. C is then in the first approximation

$$C = 1/n^2 . (\lambda_1 + \lambda_2 + \ldots + \lambda_n)(1/\lambda_1 + 1/\lambda_2 + \ldots + \lambda_n)$$

A more convenient expression is obtained for calculation if we write $\lambda_1 = l/n + \delta_1$, $\lambda_2 = l/n + \delta_2 \ldots$, viz.,

$$C = 1 + n(\delta_1{}^2 + \delta_2{}^2 + \ldots + \delta_n{}^2)/l^2 - (\delta_1 + \delta_2 + \ldots \delta_n)^2/l^2$$

A quicksilver filling of the tube has (l in meters, q in mm.^{2}) at temperature t the resistance (Strecker)

$$w = C . l/q . (1 + 0{\cdot}00090t + 0{\cdot}0000045t^2) \text{ Siemens's units.}$$

Arrangement.—The ends of the tube are fitted with corks in small tubulated cups, provided with amalgamated platinum electrodes (p. 271). The joints are made tight with collodion or gutta percha, and the whole placed in a bath.

Resistance of Widenings.—The resistance of a conductor which consists of a cylinder and a wide space limited on one side with a plane surface is about $\sigma\ 0{\cdot}80/r\pi$ greater than the resistance of the cylinder alone. In this expression σ is

the specific resistance of the conductor. This may be taken into calculation by adding $0 \cdot 80\ r_1 + r_2$ to the length of the tube where r_1 and r_2 are the radii of the two ends of the bore.

On more exact formulæ for calibration and practical rules see, among others, Siemens, *Pogg. Ann.* cx. 1, 1860; Rayleigh and Sidgwick, *Phil. Trans.* 1883, i. 173; Strecker, *Wied. Ann.* xxv. 252, 456, 1885; Benoit, *Construct. des Etalons Prototype*, etc., Paris, 1885.

72.—Resistance of an Electrolyte.

I. *With a Constant Current.*

When the resistance of a fluid which is decomposed by the current is to be measured, account must be taken of the opposing electromotive force of polarisation arising on the electrodes. The simplest method is that of substitution (**70**) in the following modified form:—

Let the fluid have the form of a column of constant section, and let an electrode be movable longitudinally in it. For this purpose either a rectangular trough filled to a certain height (Horsford) is taken, or, better, a glass tube. If the decomposition is accompanied by evolution of gas, the glass tube is bent into the form of a U, and placed with its branches upright. In the one limb is a fixed electrode, in the other an electrode which can be moved in it. The straight part of the last-mentioned limb is calibrated by measurement or weighing of successive portions of water or mercury. The fluid thus contained is included in a simple circuit with a rheostat, a galvanometer, and a galvanic cell. The position of the needle is then observed when so much of the column of fluid is included that the deflection of the needle is a convenient amount; then the one electrode is approached to the other by the length l, and such an amount R of rheostat resistance thrown into the circuit that the same deflection of the needle is produced. The resistance R is then that of the fluid between the two positions of the movable electrode. If R is given in Siemens's units, we obtain the specific conductivity k (**63**) of the fluid compared with that of mercury, as $k = l/Rq$, where q

is the area of the section in mm.^{2} and l the length in m. Compare end of **70.** q is determined by weighing (**19**). If the cylinder has the length l and volume V, $l/q = l^2/V$.

Since the conductivity of fluids varies greatly with their temperature, this must be observed, and be kept constant by placing the tube in a water-bath provided with a thermometer.

Since the polarisation is constant only with great current intensity at the electrodes, and since evolution of gas mostly occurs, platinum gauze, or a spiral wire with its plane in the section of the column of fluid, may be used instead of a platinum foil. (For the process with Wheatstone's Bridge, see Tollinger, *Wied. Ann.* i. 510.)

Resistance of Cones.—Glass tubes have usually a conical section. The resistance of a cone of length l, and with the terminal faces of area q_1 and q_2, has a resistance-capacity $\gamma = l/\sqrt{q_1 q_2}$; or if V be the volume, and we write $q_1/q_2 - 1 = \delta_1$,

$$\gamma = \frac{l^2}{V}\left[1 + \tfrac{1}{12}\frac{(q_1 - q_2)^2}{q_1 q_2}\right] \text{ or approximately } \gamma = \frac{l^2}{V}\left[1 + \frac{\delta^2}{12}\right]$$

The resistance is then γs or γ/k.

II. *With Alternate Currents* (F. Kohlrausch).

The influence of polarisation is avoided, and the resistance of an electrolyte may be measured directly, just as that of a metallic conductor, by using currents rapidly alternating in direction and of exactly equal strength, between electrodes of great capacity. Such electrodes may consist of platinised platinum foil (for "platinising" see **63**), of from 10 to 20 cm.^{2} area. The currents may be produced by a "sine inductor," consisting of a coil, within which a magnet rapidly rotates (see *Pogg. Ann. Jubelband*, p. 290). The induced currents of an induction apparatus with a rapid current interruption may also be used.

The rheostat wires must be stretched or wound double to avoid the production of extra-currents.

For measuring (or observing) the alternating currents, the electrodynamometer (**66**A, **67**A, 5) or the telephone may be employed.

The Electrodynamometer in the Wheatstone Bridge.—This

arrangement makes the measurement independent of the constancy of the current source.

The whole dynamometer is not included in the bridge, because it is not sensitive enough near the zero, but only one coil, while through the other is passed the undivided induction current.

In the figure, J is the apparatus for the production of the alternating currents, a the outer, i the inner coil of the dynamometer, F the fluid, the resistance of which is to be measured, R a known rheostat resistance of between 10 and 1000 mercury units or ohms, $A\ B\ C$ are the bridge resistances, either a stretched wire with sliding contact or two constant resistances (**71B**, to which also we refer for remarks on the reversal of the resistances, equalising the connecting wires, etc.

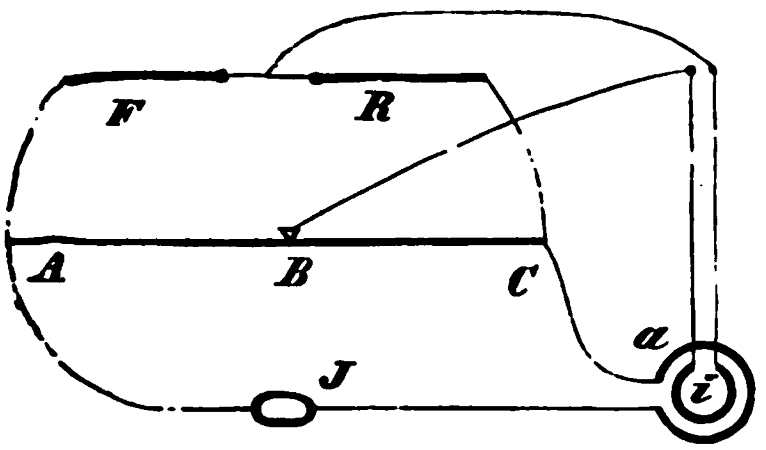


Fig. 70.

When the dynamometer shows no deflection,—

$$F : R = AB : BC$$

For the sake of increased sensitiveness, the movable coil may be hung by one wire only, while the other contact may be made by a platinised platinum wire dipping in dilute (15 per cent) sulphuric acid. To avoid fluid friction the wire must pass through the surface of the liquid at the centre of rotation.

On the vertical adjustment of the coils compare **66A**, I.

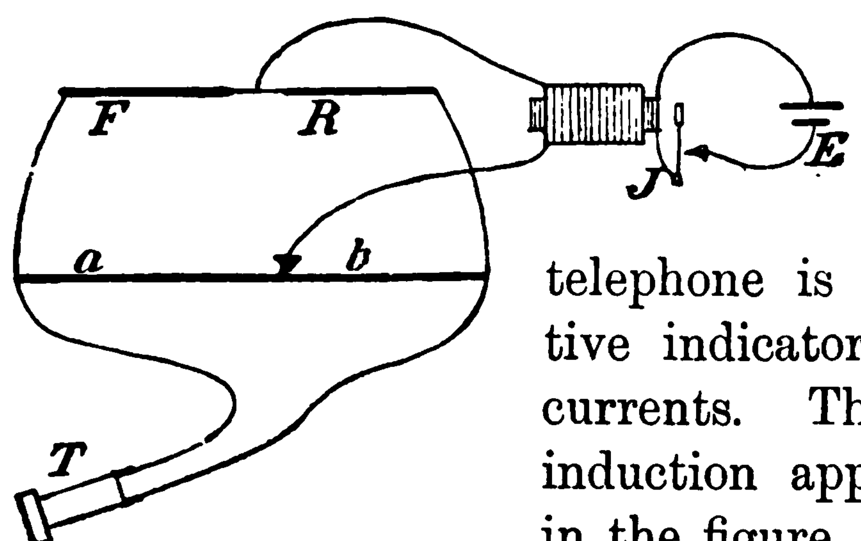


Fig. 71.

Telephone.—The Bell telephone is a convenient and sensitive indicator for rapidly alternating currents. The secondary coil of the induction apparatus J is connected, as in the figure, with the fluid resistance F, the known resistance R, and the sliding contact of the bridge ab. When the latter is so adjusted that the telephone is silent $F : R = a : b$.

Not every telephone is suitable. The sensitiveness depends on the strength of the magnet, the position and character of the diaphragm, and the resistance of the coil. Usually a cheap common telephone is suitable. The use of soft conducting wires to the telephone, removal to a sufficient distance from the interrupter, and stopping the ear not in use with cotton wool, are useful precautions to avoid interrupting noises. The telephone must be at some distance from the induction coil to prevent its excitement by induction.

The Sound-minimum.—Sometimes no position of the contact will completely abolish sound in the telephone. With good electrodes and medium resistances, say 20 to 1000 ohms, the minimum should be sharply defined. The causes of want of sharpness are various. Residues of polarisation on the electrodes may interfere, especially with small fluid-resistances. With rheostat coils of many windings, on the one hand extra-currents (self-induction), and on the other electric charges (capacity) may interfere; and the latter especially with careful double winding. It may therefore be an advantage that coils of many turns (1000 ohms and more) should be wound somewhat irregularly, so that the two halves of the wire do not always lie side by side; or that they should be wound with a single wire, of which the direction of winding is changed at the completion of each layer (Chaperon). The bath in which a badly-conducting electrolyte is immersed may also give rise to disturbance from electrical charges, which are diminished by the employment of distilled water, or, better, of petroleum.

Containing Vessels for the Electrolyte.—To make a complete absolute determination of conductivity the fluid must be contained in a vessel of which the resistance-capacity can be calculated, such, for instance, as a cylindrical tube of known area of section. If this be of 10 to 20 cm.^{2}, observations of difference can be made, as in I. (p. 316), which are independent of all assumptions. A narrow cylindrical tube may also be used connecting vessels containing the electrodes, as in the last figure, in which case the resistance-capacity of the widenings (p. 315) must be added to that of the tube. More convenient are vessels, as shown in the figure (of which the first

three are for liquids of very low conductivity, the third that of

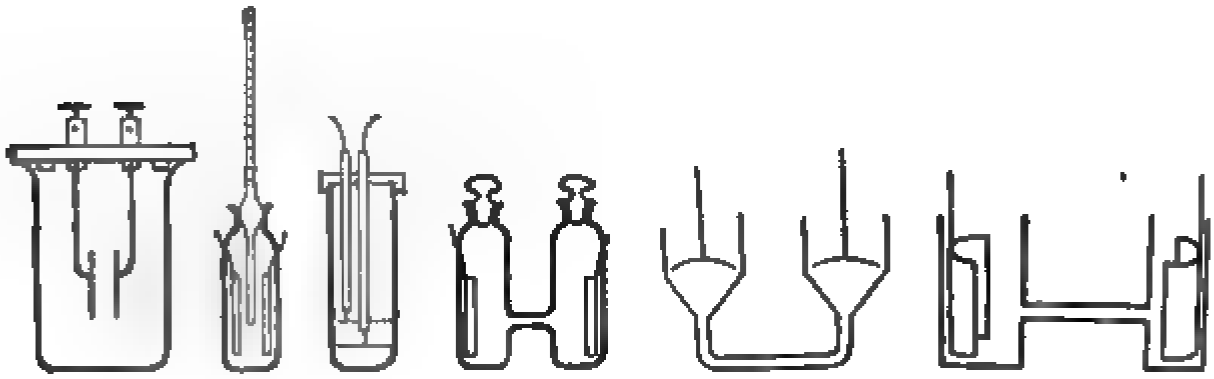


Fig. 72.

Arrhenius), of which the capacity γ is determined empirically as follows.

Empirical Determination of the Resistance-Capacity γ.— The vessel is filled with a liquid, of which the specific conductivity K has been previously determined, or with one of the following solutions, the resistance of which is sufficiently determined without an accurate quantitative analysis. According as vessels of greater or less "mercury resistance" are to be measured, a better or worse conducting fluid is chosen to fill them, so that the total resistance may have a suitable magnitude.

At temperature t the conductivity K referred to mercury at 0° is of

SULPHURIC ACID of 30·4 per cent H_2SO, Sp. Gr. = 1·224

$$10^7 . K = 692 + 11{\cdot}3\ (t - 18);$$

SATURATED SOLUTION OF COMMON SALT of 26·4 per cent NaCl., Sp. Gr. = 1·201—

$$10^7 . K = 202 + 4{\cdot}5\ (t - 18);$$

SOLUTION OF SULPHATE OF MAGNESIA of 17·3 per cent $Mg.SO^4$, (anhydrous) Sp. Gr. = 1·187—

$$10^7 . K - 46{\cdot}0 + 1{\cdot}2\ (t - 18);$$

ACETIC ACID of 16·6 per cent $C_2H_4O_2$, Sp. Gr. = 1·022—

$$10^7 . K = 1{\cdot}52 + 0{\cdot}027\ (t - 18);$$

SATURATED SOLUTION OF CALCIUM SULPHATE $CaSO_4$—

$$10^7 . K - 1{\cdot}77 + 0{\cdot}045\ (t - 18);$$

SATURATED SOLUTION OF STRONTIUM SULPHATE $SrSO_4$—

$$10^7 . K = 0{\cdot}121 + 0{\cdot}028\ (t - 18).$$

In making the last solutions, and especially that of $SrSO_4$, very pure water must be used. The saturation is soon complete if the powdered substance be used, and the presence of a moderate proportion of powder does not influence the conductivity perceptibly. The powder is repeatedly washed with the solution till a constant resistance is obtained.

To obtain the specific conductivity as referred to the legal ohm, the above numbers must be multiplied by 1·06, and, for the accurate ohm, by 1·063.

If the fluid in the vessel shows a resistance of W Siemens's units, the resistance-capacity of the latter, $\gamma = WK$. If a vessel of known capacity is at hand, that of any other is most easily obtained, by inserting the two, both filled with the same liquid, in place of F and R (Fig. 71, p. 318). The capacities are then proportional to the resistances.

Determination of the Conductivity of a Liquid.—If this has the resistance w when contained in a vessel of resistance-capacity γ, its specific conductivity $k = \gamma/w$.

In narrow vessels currents must not be used stronger, or of longer duration than necessary, to avoid evolution of too much heat.

For electrostatic methods see **84**, III.

Distilled Water may be deprived of the carbonic acid generally present by passing a current of air through it.

Compare F. K. and Grotrian, *Pogg. Ann.* cliv. 3, 1875 ; F. K., *Wied. Ann.* vi. 36, 49, 1879 ; xi. 653, 1880 ; xxvi. 168, 1885. On Influences on the Sound-minimum see also Elsas, *Wied. Ann.* xliv. 666, 1891.

73.—Measurement of the Internal Resistance of a Battery.

The methods I. to III. only give useful results with very constant batteries of considerable resistance. IV. and V. are not simple of execution. VI. is generally simpler and more exact than the rest.

I. *With the Galvanometer.*

The circuit of the battery to be examined is completed through a galvanometer (**64** to **67**), and, if necessary, a suffi-

cient additional resistance is added to reduce the deflection to a convenient amount. The current-strength J is then observed. Then, in the same circuit, an additional resistance R of known amount is included; most advantageously, such as to reduce the current-strength i, now measured, to about half its former amount. From these two observations the total resistance W of the circuit in the first observation is obtained—

$$W = Ri/(J - i)$$

From the quantity W, so calculated, we deduct the resistance of the galvanometer, previously measured, and also any additional resistance included in the first experiment, and so obtain that of the battery alone.

II. *With Galvanoscope and Rheostat.*

A circuit is formed, including an even number of cells, the galvanoscope, and a known amount of rheostat resistance, and the deflection of the needle is noted. w_1 is the total resistance of the circuit outside the battery (viz. of galvanoscope, rheostat, and connecting wires).

Secondly, the cells are arranged in pairs, with all the zincs to the same side, as shown in the annexed cut, for a battery of four cells, and the needle again brought to the same deflection as before; to effect which, a different amount of rheostat resistance will be required, and w_2 is now the collective resistance of the external circuit. Then the resistance w of the battery in the first experiment is—

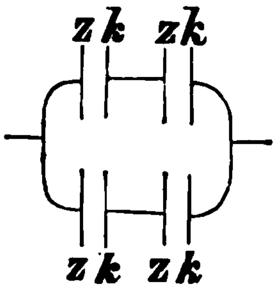


Fig. 73.

$$w = 4w_2 - 2w_1.$$

III. *With Divided Circuit* (Siemens).

The battery is closed through a galvanometer and a derived circuit. If the position of the shunt be changed so that the needle resumes its original position, the former resistance on the side of the cell was equal to that now on the side of the galvanometer, and *vice versâ.*

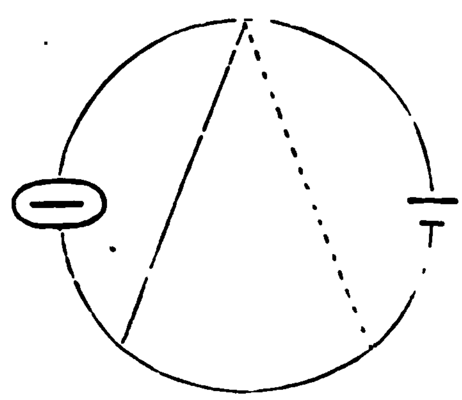


Fig. 74.

IV. *By the Method of Compensation* (v. Waltenhofen, Beetz).

Ab is a thin stretched platinum wire of known resistance, upon which are two movable contact pieces. *E* is the battery of which the resistance *W* (in which we include the resistance of its connecting wires) is to be measured. *e* is another constant battery, of less electromotive force than *E*. The batteries must be connected with *A* by their similar poles. The contact-pieces are now moved so that no current passes through the galvanoscope *G*. We denote the two resistances of the platinum wire so included by *a* and *b*.

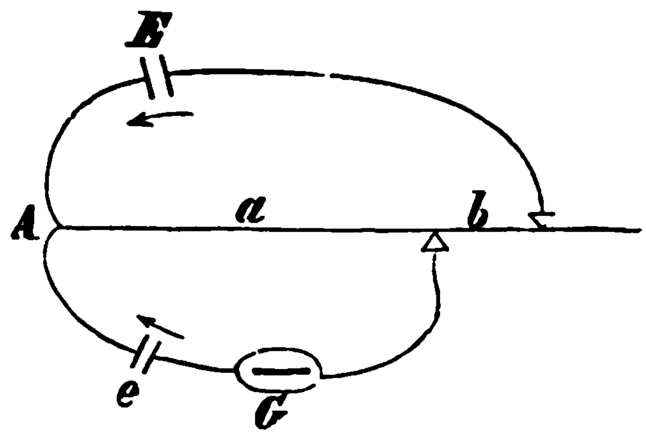


Fig. 75.

Repeating the experiment with two different resistances, a' and b', we have—

$$W = (a'b - ab')/(a - a')$$

Proof.—Since no current passes through the branch *Ge*, the circuit *A a b E* must be traversed throughout by the same current. Calling this i, we have (**63**, I. *b*) $E = (W + a + b)i$; and $e = ai$, and by division $\frac{E}{e} = \frac{W+b}{a} + 1$. Similarly we have $\frac{E}{e} = \frac{W+b'}{a'} + 1$; and therefore $\frac{W+b'}{a'} = \frac{W+b}{a}$, from which the above equation follows.

If no point of contact gives a zero current, the variable resistance must be increased or a feebler auxiliary battery employed, or a shunt-circuit of constant resistance may be added to the battery *e* (Feussner).

If the circuit be closed for a very short time only (Beetz's current-key), the method may be applied to inconstant cells. To obtain the resistance with closed circuit, a shunt-circuit is arranged to close *E*, which is broken by the current-key an instant before the connection of the batteries with the rheostat wire.

V. *In Wheatstone's Bridge* (Mance).

In Fig. 64 on p. 306 let the cell be in the branch *W*, the

galvanoscope in P_1, while the branch P_2 can be momentarily closed. If the deflection of the galvanoscope is not altered by this closing of G, the resistance of the cell is

$$W = Ra/b.$$

By a constant magnet placed near it the needle of the galvanoscope may be kept near the position of rest, and the sensitiveness so increased.

Here the resistance of the cell is measured while the current flows. The strength of the current in the cell may be measured by a tangent galvanometer included in its branch of the bridge.

VI. *By Alternating Currents.*

The resistance is most simply measured by **72,** II. Elements of not too small surface behave like ordinary conductors to alternating currents. If the dynamometer be used the cells are connected in series with their poles alternately in opposite directions. If the telephone is used, this opposition of electromotive forces is only necessary if the current is so powerful as to be injurious.

73A.—Resistance of a Galvanometer.

The resistance γ of a galvanometer may be measured like any other by **70** to **71**c; but the following methods enable us to use its own needle.

I. *Direct Closure.*

A constant battery of known and smallest possible resistance (large Daniell; on small electromotive forces see **63,** II.) is closed through the galvanometer, if necessary with added resistance, w_0 being the sum of this resistance and that of the battery, and the current-strength being J. The rheostat-resistance R is added, and the current-strength is now i. The galvanometer resistance γ is—

$$\gamma = R\frac{i}{J-i} - w_0$$

for $$(\gamma + w_0)J = (\gamma + w_0 + R)i$$

It is desirable that the second current-strength should be fully one half less than the first.

Disadvantages.—From the inconstancy of cells, the difficulty of accurately determining their resistance, and especially, because with sensitive galvanometers the resistance w_0 must be relatively too large, the method is rarely exact.

II. *Current-measure with Shunts.*

By this method it is possible, especially with sensitive galvanometers, to work with small, and, under conditions, even with somewhat changeable current-strengths, so that the constancy of the battery may be assumed. The battery of known resistance is closed through a circuit which is divided in two branches, of which one consists of the galvanometer γ, and the other of a known resistance z, which generally should not differ greatly from γ.

W is the total resistance of the undivided part of the circuit, including the resistance of the battery. It is advantageous that W should be large. Let i be the current-strength shown by the galvanometer when the resistances are w, z, and γ. With mirror-galvanometers i may be taken simply as the deflection when it is small; otherwise compare **64** to **67**.

General Case.—If W be changed to W', z to z', and to γ the resistance w is added, the current-strength i' will be produced in γ. Then

$$\gamma = \frac{i'[w(W'+z')/z' + W'] - iW}{i(W+z)/z - i'(W'+z')z'}$$

for

$$i = \frac{Ez}{\gamma(W+z) + Wz};\quad i' = \frac{Ez'}{(\gamma+w)(W'+z') + W'z'}\ \text{(p. 268).}$$

From this general formula the following methods are easily derived. The annexed figures show how the arrangement can be made with a single rheostat RR, if it is provided with some plugs with binding screws. Nos. 3, 4, and 7 especially are easily carried out.

Special Cases for Use.

(1.) W and z remain unchanged, but a resistance not greatly differing from it in amount is added to the galvanometer branch γ. The current-strength is now $= i'$. For i, W and z see above. Then

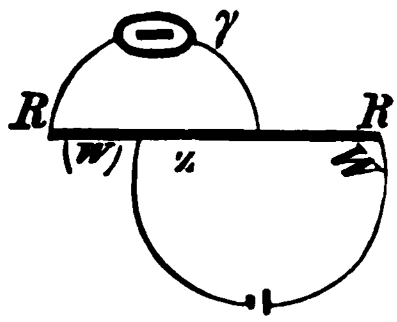


Fig. 76.

$$\gamma = \frac{i'[w(1/z + 1/W) + 1] - i}{(i - i')(1/z + 1/W)}$$

If W be very large compared to z, we have simply

$$\gamma = w \frac{i'}{i - i'} - z.$$

(2.) W and the galvanometer branch is left unaltered in the second observation ($w = o$), but z is changed to the considerably larger value z', producing current-strength i' in γ. Then

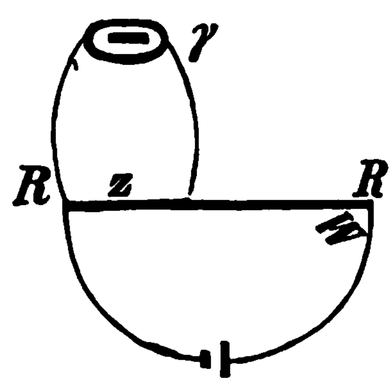


Fig. 77.

$$\gamma = \frac{i' - i}{i(1/z + 1/W) - i'(1/z' + 1/W)}.$$

If W be very large, we have—

$$\gamma = \frac{i' - i}{i/z - i'/z'}$$

(3.) W is altered to W' for the second observation, and the whole current passed through the galvanometer; that is, $w = o$ and $z' = \infty$. Then—

$$\gamma = \frac{i'W' - iW}{i(W + z)/z - i'}$$

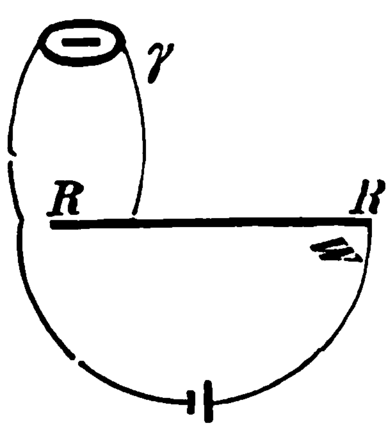


Fig. 78.

If the resistance in the undivided circuit remains unaltered during both observations ($W = W'$), then—

$$\gamma = \frac{i' - i}{i(1/z + 1/W) - i'/W}$$

and if W is very large, $\gamma - z(i' - i)/i$.

In general it is desirable that one current-strength should be about half the other.

The employment of a commutator to the battery is, of course, advantageous.

The methods 1 to 3 are especially applicable to mirror galvanometers of not too large resistance, where ordinary methods fail.

III. *Division of Circuit with Equal Current-Strength.*

In order to apply the methods to galvanoscopes, which allow of no actual measurement, the resistances must be so regulated that the current-strengths are the same in both observations ($i = i'$).

In this case, universally (see above)—

$$\gamma = z\frac{w(W' + z') + z'(W' - W)}{Wz' - W'z}$$

and we have choice of the following methods.

(4.) The resistance W of the undivided circuit remains constant ($W = W'$). To the galvanometer branch γ a resistance w is added, which considerably reduces the current-strength (say by one half). z is then increased to z' till the originally current-strength is restored. Then—

$$\gamma = \frac{wz}{z' - z}\left(1 + \frac{z'}{W}\right)$$

and for very high W, simply $\gamma = wz/(z' - z)$.

In carrying out the method according to Fig. 76 (p. 326), the exact resistance which makes $i' = i$ must if necessary be interpolated from two neighbouring resistances and current-strengths (5).

(5.) z remains unchanged, w is added to γ, and W diminished to W' till the original current-strength is restored, when—

$$\gamma = w\frac{W' + z}{W - W'} - z.$$

On interpolation see (4).

(6.) The galvanometer branch remains unaltered in both

observations ($w = o$). If z and W give the same current-strength as z' and W', then—

$$\gamma = \frac{W - W'}{W'/z' - W/z}$$

Cf. Fig. 77, and what is said on interpolation in (4).

(7.) With the resistance W of the main current, and the branch resistance z, we give the galvanometer the same deflection as with the larger resistance W', and without branching ($w = o, z' = \infty$, Fig. 78). Then—

$$\gamma = (W' - W)/W$$

IV. *In the Wheatstone Bridge* (Thomson).

The galvanometer is included in one of the four branches of the bridge (Fig. 59, p. 303), for instance, in d. The bridge itself G is formed simply of a connecting wire and contact breaker. When the deflection of the galvanometer remains unaltered on closing and opening the bridge, $a : b = c : d$. If the deflection of the galvanometer be too large, it may be reduced by approaching to it a suitable magnet. In practice, the trials before the proper position is found consume considerable time.

V. *By Damping.*

According to **71**c. If the logarithmic decrement of the needle is λ_0 with direct closed circuit, λ when the known resistance R is included, and λ' with the circuit open, the resistance of the galvanometer is $\gamma = R(\lambda - \lambda')/(\lambda_0 - \lambda)$.

74.—Comparison of Two Electromotive Forces (*Potential Difference*).

In order to measure an electromotive force, we may compare it with that of some known constant element (Daniell, Clark). If the latter is given in absolute measure (volts), (**63**, II., and **76**), that of the battery to be compared may be expressed in the same terms.

In judging of the measurements we must remember that no galvanic cell is quite constant in its electromotive force. Independently of the variations arising from the time the current has passed, the electromotive force of all cells decreases with increasing current. In cells with large plates depositing copper from concentrated solutions or with strong nitric acid, the diminution of electromotive force is not perceptible with moderate current-strength. Elements with weak or exhausted solutions, especially of chromic acid, and "inconstant" elements (*e.g.* Smee, Leclanché), may show with powerful currents a value many times smaller than when "compensated" or with a quite weak current.

I. *Comparison with Galvanoscope and Rheostat.*

A circuit is formed, including a rheostat, a galvanoscope, and an electromotive force E. If necessary, as much extra resistance is intercalated as will reduce the deflection to a convenient amount. The second electromotive force is then substituted for the first, and the current brought to the same amount as before by means of the rheostat. Calling the total resistance in the first experiment W, and that in the second w, we have—

$$E : e = W : w.$$

W and w are in each case the resistances of the rheostat and that of the remainder of the circuit taken together, including the internal resistance of the battery itself. If, however, the resistance of the rheostat be very large compared to that of the remainder of the circuit, which may always be made the case by employing a very sensitive galvanometer, the latter may be neglected, or at least may be roughly estimated.

II. *Comparison by the Galvanometer.*

If two electromotive forces, E and e, produce in circuits of resistance, W and w, the current-strengths J and i, then—

$$E : e = JW : iw$$

The method becomes very simple and independent of all

measurement of resistance if we connect the batteries through a sensitive galvanometer (**66, 75, 77**) and a very large constant resistance, so that that of the battery may be neglected. The current-strengths being J and i, we have simply—

$$E : e = J : i$$

III. *Poggendorff's Method of Compensation.*

The only methods applicable to inconstant elements, of which the electromotive force varies with the current-strength, are those which bring the current to zero by opposing an equal electromotive force.

Exact compensation is often tedious, since during the testing currents are produced, of which the influence on the electromotive force of the battery lasts for a time. Hence in seeking the correct compensation the closures should be as short as possible, but in the final observation should be maintained long enough to show whether compensation has been actually reached.

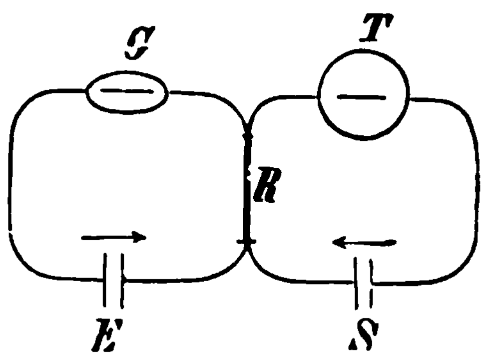


Fig. 79.

In the left division of the circuit is the galvanoscope G, and one of the electromotive forces E to be compared; in the right, the auxiliary battery S, and the galvanometer T. E and S are so placed that their similar poles are turned towards each other. R is a rheostat.

As much rheostat resistance R is intercalated as will cause the current in G to vanish, and the current-strength J in T is observed.

The other electromotive force e is now substituted for E, and the current in G again reduced to zero by the rheostat-resistance r. The current-strength i in T is now observed.

Then—

$$E : e = JR : ir$$

It follows from (**63**, I. b) that $E = JR$ and $e = ir$ when the current in the branch G is zero.

It is sometimes a convenience to intercalate an additional resistance in the branch S.

IV. *Bosscha's Method of Compensation.*

The electromotive force e of a possibly inconstant element is to be compared with E, that of a stronger constant battery (one or more Daniell's cells). Let a and b be the lengths of rheostat-wires with good sliding contacts (say stretched wires measured from the connected ends to the contacts) which are required in order that no current may pass through the galvanoscope G. In a second experiment this is effected by a' and b'. Then—

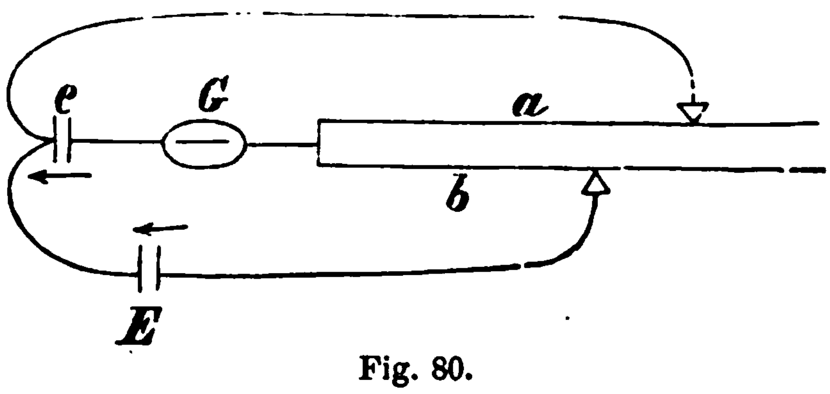


Fig. 80.

$$\frac{E}{e} = 1 + \frac{b - b'}{a - a'}$$

The experiment may also be arranged with only one wire with two sliding contacts, as in the figure, when the same relation holds. In the first case the resistance of both contacts, and in the second that behind b, must be constant. Small tubes filled with mercury and sliding on bright platinum wire are serviceable.

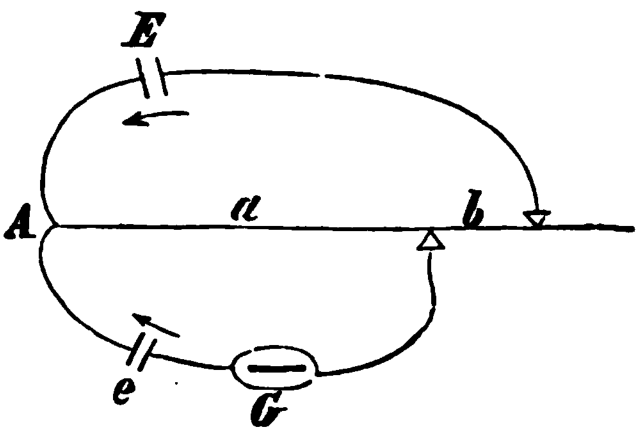


Fig. 80a.

Compare **73,** IV. for the proof and conditions under which the method is practicable.

V. *Dubois-Raymond's Method of Compensation.*

If, in the last figure, the length $a + b$ is constant and E is unchanged, we have the desired electromotive force e simply proportional to the length a, therefore $e = C \,.\, a$.

For if W is the resistance of the cell E and its connecting wires—

$$e : E = a : (W + a + b)$$

The factor C may be obtained once for all by using for e a known element (Clark or Daniell's cell).

In order that the methods IV. and V. may be practicable, it is necessary that at least one resistance must amount to $a = We/(E-e)$. If a does not amount to this, a stronger or larger cell E must be used.

VI. *Clark's Method of Compensation.*

Two inconstant electromotive forces may be compared directly by means of two galvanoscopes G_1 and G_2 and a more powerful constant auxiliary battery E. Let R be a rheostat and a b a wire with sliding contact. Let $E > E_1 > E_2$. By unplugging resistance in R, and simultaneous adjustment of the sliding contacts, the current in G_1 and G_2 may be brought to zero. Then, obviously

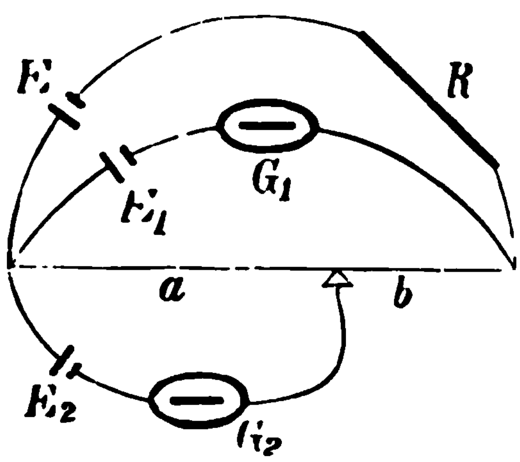


Fig. 81.

$$E_1 : E_2 = (a + b) : b$$

A Resistance Box with Plugs may also be used in IV., V., and VI., for the resistances $a + b$, by connecting firmly the wires from E to the ends of the rheostat, and with the wires from e and G touching the two ends of so much of the resistance that the current in G disappears. The last-named contacts need not be free from resistance.

For electrostatic methods see **84.**

75.—Siemens's Universal Galvanometer.

This instrument may be used as a sine-galvanometer (**65**); it further contains arrangements for determining resistances, and for the comparison of electromotive forces by the bridge methods.

The accompanying figure represents, in a diagram, the parts and connections of the universal galvanometer. G is the galvanometer coil; R resistances of 1, 10, 100, or 1000 ohms,

which are thrown into the circuit by removing the plugs; a b the circular stretched platinum wire. I., II., III., and IV. are binding-screws, of which III. and IV. can be connected with each other by means of a plug. In the later instruments there is also a binding-screw V., with a contact-key to II., and which is used in place of the latter for momentary closure; for the same purpose an easily made contact with II., such as a plug in a conical hole, and fitted with a binding-screw may be used. Finally, C is the movable contact which can be placed on the platinum bridge-wire (the connection of this with I. is in reality made under the instrument).

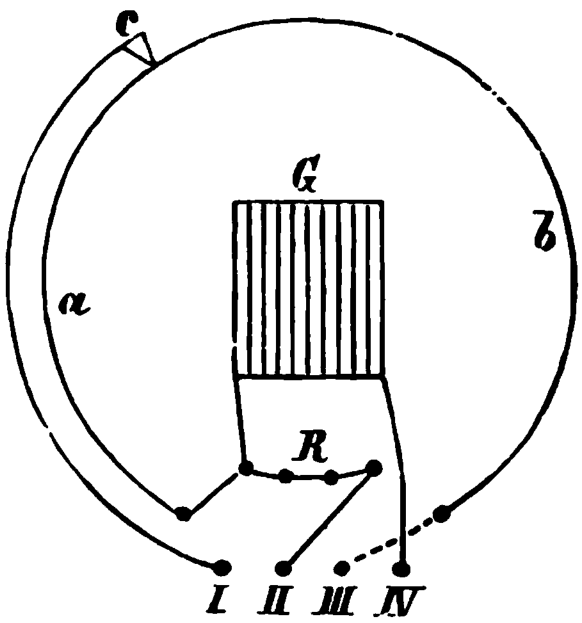


Fig. 82.

I. The instrument is used as a sine-galvanometer by simply connecting II. (or V.) and IV. with the circuit. By removing a plug from R a resistance may be at the same time inserted. The divisions of the platinum wire, which are numbered as degrees, are used as the graduation of the circle.

II. In order to compare resistance W with one of the resistances of R, the screws I. and II. (V.) are connected with a cell, II. and III. with W, and the plug inserted between III. and IV. It will be easily seen that the connections are then essentially as in the figure in **71**B, II.

That position of the sliding contact C is now sought at which the placing of C on the platinum wire communicates no deflection to the galvanometer-needle. Then $W : R = b : a$.

R is so chosen that a and b are as nearly equal as possible. $b + a = 300$; the zero of the scale is in the centre. A table facilitates the calculation.

III. For the comparison of electromotive forces after the compensation methods (**74**, IV. and V.), the plug between III. and IV. is removed, but the plugs of R remain in. The weaker electromotive force e is inserted between the screws I. and IV., the stronger and constant comparison cell E between II. (V.)

and III., their similar poles being connected to I. and III. Then the length a is sought at which the galvanometer-needle is not moved by a momentary closing of the current.

If e be inconstant, its circuit must be last closed, which may be done either with the key or with the screw I. If the resistance W_0 of the battery E is known, $e : E = a : (a + b + w_0)$.

If e' be a second battery, for which by comparison with E the length a' is found,

$$e : e' = a : a'$$

76.—Electromotive Force in Absolute Measure.

According to Ohm's Law, the absolute measure of an electromotive force E is given by the current J which it produces through the resistance W, as $E = WJ$.

W and J measured in absolute electromagnetic units [cm. g.] give the force in units of the same system; that is, in $[\text{cm.}^{\frac{3}{2}}\ \text{g.}^{\frac{1}{2}}\ \text{sec.}^{-2}]$; ohms and amperes give volts (compare **63**, I., and App. 20).

1 volt $= 10^8\ [\text{cm.}^{\frac{3}{2}}\ \text{g.}^{\frac{1}{2}}\ \text{sec.}^{-2}]$; 1 Siem. unit $= 0{\cdot}9434$ legal ohms.

I. *Direct Measurement.*

If the resistances of the current-source w_0 and of the galvanometer γ are known, so that $w_0 + \gamma + R$, the total resistance of the circuit, including the added rheostat resistance R, can be calculated, the current-strength J gives the electromotive force as above by the equation

$$E = (w_0 + \gamma + R)J$$

With sensitive galvanometers w_0 may generally be neglected, and even γ frequently need not be considered (compare also **77**).

The method can be directly employed even when the circuit of current-source (battery or dynamo) is otherwise closed, as, for instance, on dynamos of ordinary winding, which only have potential when they run closed. The galvanometer γ, and sometimes the rheostat resistance R, then form a shunt-

circuit, and $E=(\gamma+R)i$ is the so-called "terminal potential difference" (**76**A and **77**A).

Voltmeters are galvanometers so graduated as to give directly the product resistance × current-strength.

Compare also the employment of the electrometer in **84**.

II. *Ohm's Method.*

By measurement of two current-strengths the battery and galvanometer resistance is eliminated. The circuit is closed through a rheostat and galvanometer, and the current-strengths i_1, i_2 are observed with the resistances R_1, R_2. Then

$$E = i_1 i_2 \frac{R_1 - R_2}{i_2 - i_1}$$

It is conducive to accuracy of result to so choose resistances that the strength of one current is about half that of the other. 35° and 55° are the best deflections for the tangent-compass.

The method is limited to "constant" cells; but the *E.M.F.* of all batteries is diminished by strong currents (p. 329). The method is inapplicable to dynamos.

III. *Poggendorff's Method.*

By the combination shown in Fig. 79, p. 330. When the current in the galvanoscope G is reduced to zero by intercalation of resistance W in the rheostat, if $J=$ current-strength in T, the electromotive force of the battery E is

$$E = WJ$$

This method is universally applicable. (Compare directions, p. 330.)

76A.—Potential Difference in Closed Circuits.

To find the difference of potential which exists between two points in a circuit, the method given under **76**, I. is adopted. A sensitive galvanometer, with large added resistance, is placed in a derived circuit joining the two points. If W be the total resistance, and i the current-strength in the

derived circuit, the difference of potential for very large W is simply $P = iW$.

If the remaining resistances cannot be neglected as compared to W, a correction must be made. If w be the resistance of the principal circuit between the two points, and w_0 the resistance of the remainder of the circuit, the tension before adding the shunt was

$$P = i\left(W + \frac{w_0 w}{w_0 + w}\right):$$

i measured in am. and w in ohms gives P in volts.

The Terminal Potential Difference (Ger. *Klemmspannung*).—This is the difference of potential between the two terminals of the current-source during the passage of the current, and is determined as above. P is only identical with the total *E.M.F.* in case of very great external resistance w, otherwise, if w_0 be the internal resistance of the current-source,

$$E : P = (w + w_0) : w$$

77.—Siemens's and Halske's Torsion Galvanometer (Frölich).

Current-Measure.—In each measurement the needle is brought back to zero, and parallel with the coils of the multiplier, by turning the torsion head through an angle a proportional to the current-strength i, which is therefore

$$i = C \,.\, a$$

C is determined by the silver voltameter (**68**, I.), with a Clark or other constant element (**68a**), or by comparison with a normal galvanometer. Compare **69**. Thermo-elements may also be employed as a control (**69**, III.).

The two patterns of the instrument made by Siemens and Halske have $C = 0{\cdot}001$ and $C = 0{\cdot}0001$ am., and the resistances of their galvanometer coils are 1 and 100 ohms respectively. Strong currents are measured by shunts. The shunt-resistance z makes the reduction-factors $C = 0{\cdot}001\,(z + 1)/z$ and $C = 0{\cdot}0001\,(z + 100)/z$ am. Round numbers are therefore attained by taking $z = 1/9$, $1/99$, etc., and $z = 100/9$, $100/99$, etc.

Measurement of Potential.—The addition of a resistance of R ohms to the instruments produces values for 1 scale-division of $0{\cdot}001(R+1)$ and $0{\cdot}0001(R+100)$ volts respectively. Resistances of $R=9$, 99, 999 ohms and $R=900$, 9900, 99900 ohms may be added, producing scale-values 1 div. = 0·01, 0·1, 1 volt, and 0·1, 1, and 10 volts. Changes of temperature influence the accuracy of the measurements. To avoid the heating effect of the current as far as possible, the circuit should be closed only during the measurement.

On the corrections required in the measurement of potential difference (*P.D.*), compare **76**A.

The torsion galvanometer is independent of terrestrial magnetism if the needle be in the plane of the meridian. On the other hand, variations in the magnetism of the needle by time or the action of too strong currents alter the value of the constant, which must therefore be redetermined from time to time.

For fuller details see v. Waltenhofen, *Zeits. für Elektrotechn.* p. 154, 1886. On Strong Currents, W. Kohlrausch, *Centr. Bl. für El. Techn.* p. 813, 1886.

77A.—MEASUREMENTS ON DYNAMOS.

In the first place, we must distinguish between continuous and alternating-current machines.

According to the relation of the armature, the actual current-source of the machine, to the other parts of the circuit, the continuous-current machines are divided into—

1. *Ordinary Series Machines.*—The armature is in series with the coils of the electromagnet and the outer circuit. With increasing external resistance, the *E.M.F.* diminishes to zero. The terminal potential difference has a maximum with a certain external resistance.

2. *Shunt Machines.*—The magnet coils and the external circuit are connected in parallel circuit. The terminal *P.D.* is zero with small external resistance and increases with increase of the latter to a limiting value.

3. *Compound Machines of Constant E.M.F.*—The electromagnet has a double winding. One set of coils is in series with the armature and external circuit, while the other is a shunt to the latter.

Suitable proportions make the terminal *P.D.* almost independent of the external resistance.

The shunt coils of 2 and 3 are usually provided with adjustable resistances.

I. *Current-Strength.*

On the measurement of very powerful currents with the tangent galvanometer, or reflecting and torsion galvanometers with shunts, or with the technical "ammeters," compare **64**, III., **66**, **66a**, **67a**, **77**. In using the mixed gas voltameter, **68**, III., it must be noted that it introduces an opposing *E.M.F.* of about 2·5 volt into the circuit.

The fluctuations which always occur in dynamo-currents must be rendered as harmless as possible by the use of powerful damping on the galvanometer. To obtain a mean value the readings should be made at a series of regular intervals, say, every 5 or 10 seconds. On the regulating of currents by accumulators compare p. 275.

II. *Resistance.*

On methods of measurement see **70** to **71c**. Since for technical purposes the practical resistances are those of the machine and circuit during use (that is, in warmed condition), the measurements are taken immediately after prolonged action.

Most conveniently, a measured current from accumulators of the same order of magnitude as that of the machine is passed through the circuit, and the terminal potential difference is measured. The armature must be fixed if the field magnets are included in the circuit. The brushes must come well down on the commutator, which must be freshly cleaned with emery (*cf.* **71**, II. 2).

On determination of the resistance of lamps during use see **71**, II.

III. *Electromotive Force or Potential.*

Here many of the ordinary methods are inapplicable, since the *E.M.F.* lasts only during the passage of the current, and

is dependent on its strength; and can therefore neither be measured by compensation nor the intercalation of a great resistance. On the other hand, the terminal *P.D.* or external and disposable *E.M.F.* is directly measurable by **76A**. A properly adjusted torsion galvanometer (**77**) has the advantage that no calculation is required.

In series machines the total *E.M.F.*, E, is

$$E = P(w + w_0)/w \text{ or } P + w_0 i$$

where w_0 is the internal, and w the external resistance, i the current-strength, and P the terminal potential difference.

Current-Work in Unit of Time or Current-Power is the product of electromotive force or potential and current-strength. Volt and ampere units give this quantity in volt-amperes or watts (as, for instance, the power of a water-power is *weight of water/sec.* × *height of fall*). According to App. 22:

1 volt-ampere or watt $= 0{\cdot}102$ *kg. wt.* × *m/sec.* $= 0{\cdot}00136$ horse-power. 1 horse-power $= 746$ watts.

The total electrical work of series machines is $= Ei$, the external or useful work $e = Pi$. The ratio between the two or P/E may be called the *electrical efficiency* of the machine.

Efficiency is the proportion of the useful electrical effect to the mechanical work L which is required to drive the machine; that is Pi/L. If a machine, for instance, for a consumption of 1 horse-power yields 600 watts outside work, corresponding to $600 \times 0{\cdot}00136 = 0{\cdot}82$ horse-power, the efficiency is $0{\cdot}82$, or 82 per cent.*

Alternating Currents.

The *average external work/sec.* is, according to the previous definition, the sum of current work, $Pi\,dt$ in all the single time-infinitesimals dt which make up 1 sec., while P is the terminal poten-

---

* *Gross or "Commercial" efficiency* is the ratio of total electrical effect; *net efficiency*, that of the useful or external electrical effect to the mechanical power required to produce it; while *electrical efficiency* is the ratio of the *net* to the gross.—*Tr.*

tial difference. If t be the period of the current (or also a great period of time as compared to the latter), we have

$$e=\frac{1}{t}\int_0^t Pidt, \text{ or } \frac{w}{t}\int_0^t i^2dt, \text{ or } \frac{1}{tw}\int_0^t P^2dt$$

where w denotes the external resistance. The second expression is only valid when no external electromotive forces are present, and the third generally in the absence of inconstant external electromotive forces, which, however, are frequently present on account of self-induction. Through these the maxima of terminal *P.D.* and current-strength do not usually coincide, but the current-curve falls behind that of potential (difference of phase). Then the mean of Pi is in general less than i^2w or P^2/w, and the first integral similarly smaller than the others.

The actual energy is therefore measured in the following way:—The fixed coil of an electrodynamometer, of which the self-induction must be very small, is traversed by the current i, while the movable coil with a great resistance joins the points between which the energy is to be measured (*e.g.* the terminals of the machine when entire P is to be measured). e is then obtained by multiplying the deflection by a factor determined with a constant current. The deflection is obviously $= Ci'i = C/w'Pi$, where C is the reduction-factor of the dynamometer, i' the current-strength and w' the (great) resistance of the derived circuit; compare **66A**, **69**, **76A**.

Mean Current-Strength and Mean Potential.—If these magnitudes are always taken as positive, $\frac{1}{t}\int idt$ and $\frac{1}{t}\int Pdt$ may be looked on as the mean values. If, however, the mean values are to be so defined that the energy can be calculated from them, as in the case of a constant current (see above), the expressions should be chosen

$$\sqrt{\left(\frac{1}{t}\int i^2dt\right)} \text{ and } \sqrt{\left(\frac{1}{t}\int P^2dt\right)}$$

In these definitions the mean current-strength is measured by an electrodynamometer (**66A**).

On construction, theory, and measurement of dynamo-machines see among others A. Kittler, *Handbuch d. Electrotechnik*, i. 297 and

361 and *seq.*, 1886; ii. 1, 1889. On measurement of work, *ib.* i. 326. Also Frölich, *Die Dynamo-maschine*, Berlin, 1886.

IV. *Photometry of Electric Lamps.*

On photometry compare **47A**. In this case measurements of light and current-strength or potential must be carried out simultaneously. Either a determined current or potential is given, and the amount of light measured which it produces, or the photometer is set to the light which the lamp is supposed to give, and the current is regulated with the rheostat, etc., till that light is actually produced. For glow lamps the latter method is generally employed.

Arc lamps cannot well be compared directly with a normal candle. As an intermediate, a large petroleum lamp, lighted half an hour before use, is compared with the standard candle. For the sake of convenience this lamp may be regulated to a round number as 20, 50, 100 candles.

In order to measure the light radiated by the electric lamp in various directions, a mirror is conveniently employed which is inclined at an angle of 45° to the bench of the photometer, and capable of rotation on an axis parallel to the latter. The mirror, of course, reflects only a portion of the light which falls upon it, and the loss must be specially determined.

Cf. Krüss and Voit, *Bericht d. Münchener El. Ausstellung*, ii. p. 76; v. Hefner-Alteneck, *El. techn. Zeits.* 1883, 445; Leonh. Weber, *ib.* 1884, 176; Möller, *ib.* 1884, 370, 405; Krüss, *Die el. Photometrie.*

77B.—Galvanic Determination of the Horizontal Intensity of Terrestrial Magnetism or of a Magnetic Field.

I. *With Voltameter and Tangent-Compass.*

Let the same current pass through a tangent galvanometer of mean radius R and number of windings n, and a voltameter of the electrochemical equivalent A in relation to cm. g. In the time t sec. the quantity m is separated, while ϕ is the mean deflection of the galvanometer. Then according

to **64** and **68**, the current-strength i is on the one hand $= tan\,.\,\phi RH/2\pi n$ (cm. g.), and on the other $= m/At$, and the horizontal intensity H is

$$H = \frac{m}{At}\,\frac{2\pi n}{R}\,\frac{1}{tan\,\phi}\ \text{(cm.g.)}$$

A is in this case ten times greater than for amperes, viz. $= 11{\cdot}18$ mg. silver, $3{\cdot}280$ copper, etc. If the suspension thread has torsion, R must be multiplied by the factor $(1+\Theta)$. As regards corrections for length of needle and section of coils see **64**, II.

II. *With the Bifilar Galvanometer and Tangent-Compass* (W. Weber).

Let the current traverse a bifilar galvanometer (**67**) of directive force D (**53**), and coil-surface f (**83**), and a tangent galvanometer (see above). The simultaneous deflections being a for the bifilar and ϕ for the tangent galvanometer, the horizontal intensity H is obtained from the following equation—

$$H^2 = \frac{D}{f}\,\frac{2\pi n}{R(1+\Theta)}\,\frac{tan\ a}{tan\ \phi}$$

With a small bifilar this method may also be applied to strong magnetic fields, *e.g.* between the poles of an electromagnet (Stenger, *Wied. Ann.* xxxiii. 312, 1888).

Current-strength.—The combination of tangent and bifilar also gives current-strength in absolute measure from

$$i^2 = D/f\,.\,R(1+\Theta)/2\pi n\,.\,tan\ a\ tan\ \phi.$$

The current is commuted in both instruments. On corrections of the tangent galvanometer compare **64**, II.

The expressions are obtained by eliminating i or H from the combined equations of the single instruments (**64** and **67**).

Cf. F. K., *Pogg. Ann.* cxxxviii. 1, 1869.

III. *With the Bifilar Galvanometer and a Magnetic Needle* (F. K.)

A short needle is suspended at the distance of a cm. north or south from the centre of the bifilar coil, and at the same

height. The current which produces the (small) deflection α of the bifilar galvanometer deflects the needle through the angle ψ. r is the mean radius of the bifilar coil and Θ the coefficient of torsion of the magnetic needle (**55**).

The horizontal intensity is derived from

$$H^2 = \frac{D}{a^3(1 - \frac{9}{8}r^2/a^2)(1+\Theta)} \frac{\sin\alpha}{\tan\psi}$$

the current-strength from

$$i^2 = \frac{a^3 D}{f^2}\left(1 - \frac{9}{8}\frac{r^2}{a^2}\right)(1+\Theta)\frac{\tan\alpha\tan\psi}{\cos\alpha}$$

The proportion of the length of needle, and breadth and thickness of windings to the distance a is assumed to be so small that its square may be neglected as compared to 1.

In order to avoid uncertainty in the determination of a, the magnetometer is placed first north and then south of the bifilar, and a is taken as half the distance of the two positions of the suspending thread, while the mean of the two deflections is taken as ψ. As a matter of course the observations are made with the current in both directions.

Compare also **60**A.

Proof.—If the area of windings of the coil $=f$, the current i produces a deflection α given by the equation $D \,.\, \sin\alpha = fiH \,.\, \cos\alpha$. The deflection ψ of a short needle at the distance a from the middle point of the hanging coil, itself deflected through the small angle α, is given by

$$H(1+\Theta)\sin\psi = \frac{fi\cos\alpha}{a^3(1 - \frac{9}{8}r^2/a^2)}\cos\psi$$

from which the above expressions are obtained.

Observations in the First Arrangement (**59**).—The magnetometer may be placed east and west, instead of north and south of the bifilar, when

$$H^2 = \frac{2D}{(a^2+r^2)^{\frac{3}{2}}(1+\Theta)} \frac{\sin\alpha}{\tan\psi}$$

On the carrying out of the method, and on some corrections, compare F. K., *Wied. Ann.* xvii. 737, 1882.

77c.—Electromagnetic Rotation of Light (Verdet's Constant).

Let a beam of polarised light traverse a "magneto-optically active" body of the length l in the direction of the lines of force of a magnetic field H (App. 16). The angle of rotation of the beam is then (Faraday, Verdet)

$$a = C \, . \, Hl$$

C is the magneto-optical or Verdet's constant of the body. The rotation takes place in the direction of the electric current which produces the magnetic field by flowing round it.

On the measurement of a see **46**, I. 2 to 5. The magnetic field is produced between the broad poles of an electro-magnet with the smallest possible perforation; or for exact measurements, in a long, narrow, evenly wound bobbin. In the first case H is determined empirically by **77b**, II., or **81b**; in the second $H = 4\pi ni$ (cm. g.), if i is the current-strength in (cm. g.), n the number of windings on 1 cm. of the bobbin (p. 275; App. 19).

For sodium light at 18° (Arons, H. Becquerel, Bichat, and de la Rive, Gordon, Rayleigh) in

Carbon Disulphide.	Water.
$C = 0{\cdot}042'$	$0{\cdot}013'(cm.^{-\frac{1}{2}}g.^{-\frac{1}{2}}sec.)$

C diminishes with increasing temperature, in the case of CS_2 about $\frac{1}{900}$ per 1° C. It is approximately inversely proportional to the square of the wave-length λ of the light; more exactly $C = a/\lambda^2 + b/\lambda^4$.

Current Measurement.—Strong currents may be approximately measured by the rotation in CS_2, etc., in a bobbin by the above formula.

78.—Laws of Motion of a damped swinging Magnetic Needle.

Let

K be the moment of inertia of the needle (**54**);

D the directive force (App. 9), which for a single needle $= MH(1 + \Theta)$;

p the damping-constant, that is, the factor by which the angular velocity at any time must be multiplied to give the moment of rotation opposing the movement;
u_0 the angular velocity at the passage over the position of rest;
a the deflection which would take place without damping;
$a_1\ a_2\ a_3\ \ldots$ the deflections which take place with damping;
$k = a_1 : a_2 = a_2 : a_3 =$ the ratio of damping;
$\lambda = log.\ k$, the Briggs logarithmic decrement;
$\Lambda = log.\ nat.\ k = 2{\cdot}3026\lambda$, the natural logarithmic decrement, (which for small dampings $= k - 1$);
T the period of oscillation;
τ the period of oscillation which would occur without damping;

Then the following relations hold good—

$$\frac{P}{K} = 2\frac{\Lambda}{T} \qquad (1) \qquad\qquad \frac{K}{D} = \frac{\tau^2}{\pi^2} = \frac{T^2}{\pi^2 + \Lambda^2} \qquad (2)$$

and $$T = \tau\sqrt{1 + \Lambda^2/\pi^2} = \tau/\pi\,.\,\sqrt{\pi^2 + \Lambda^2} \qquad (3)$$

For feeble dampings, as π^2 nearly $= 10$, and Λ is $k - 1$, we may write $T = \tau(1 + \frac{1}{20}(k - 1)^2)$. A damping of a few per cent does not noticeably influence the period of oscillation. Compare Table 21B.

If u_1 is the velocity at the first return to the position of equilibrium, then

$$u_0 = ku_1 \qquad (4)$$

and $$a = a_1 k^{1/\pi\,.\,tan^{-1}\,\pi/\Lambda} \qquad (5)$$

Lastly, the velocity at the beginning is obtained as

$$u_0 = \pi/\tau\,.\,a = \pi/\tau\,.\,a_1 k^{1/\pi\,.\,tan^{-1}\,\pi/\Lambda} \qquad (6)$$

The exponential factor up to $k = 2$, that is, to $\lambda = 0{\cdot}3$, or $\Lambda = 0{\cdot}7$, may be written sufficiently exactly as $1 + 1{\cdot}160\lambda$; and, for slight dampings, as $\sqrt{k}$. Compare also for this and for $\sqrt{\pi^2 + \Lambda^2}/\pi$ Table 21B.

If the damping-constant p equals or exceeds $2\sqrt{KD}$, no further oscillations occur, but the needle approaches its position of equilibrium aperiodically.

On the decrease of the ratio of damping with the arc of oscillation see K. Schering, *Wied. Ann.* ix. 471, 1880.

Damping, Galvanometer-constant, and Resistance.—If the damping is caused by a multiplier, a close relation exists

between the logarithmic decrement and the galvanometer constant. Indicating by q the moment of rotation exerted on the needle by a unit current in the multiplier, that is, the so-called "dynamic galvanometer constant," qu will be the *E.M.F.* which is produced in the multiplier by a velocity u of the needle. If w be the resistance of the circuit in absolute measure, the current produced will be qu/w, and from this arises the damping moment of rotation, which will have the magnitude $q \cdot qu/w = uq^2/w$. Therefore q^2/w is the constant of damping, which we have called above p, and which, according to equation 1, equals $2K\Lambda/T$. Therefore,

$$q^2/w = 2K\Lambda/T \tag{7}$$

We may therefore determine q or w from K, Λ, and T, if w or q is known. Further, let

G denote the "static galvanometer constant"; that is, the factor by which a current-strength must be multiplied in order to produce the corresponding permanent deflection ϕ (or its tangent) without torsion of suspending fibre, and in a field of unit terrestrial magnetism. Let M be the magnetism of the needle (which for an astatic pair must be taken as the difference of magnetism of the two needles). Since then $qi \cos \phi_1 = M \sin \phi_1$, and $iq/M = \tan \phi_1 = G \cdot i$,

$$q = GM \tag{8}$$

Finally, as in **64** *et seq.*, let—

C be the ordinary reduction-factor of the galvanometer, which, multiplied by the deflection ϕ (or its tan), gives the current strength in absolute measure, for the terrestrial magnetism H and the coefficient of torsion Θ. Then—

$i = C \tan \phi$, while $\tan \phi_1 = Gi$, so $GC = \tan \phi_1 / \tan \phi = H(1 + \Theta)$.

Then $G = H(1 + \Theta)/C$, and, finally,

$$q = MH(1 + \Theta)/C. \tag{9}$$

As in fact, part of the damping arises from the resistance of the air, it is necessary in eq. 7 to substitute $\Lambda - \Lambda'$ for Λ, where Λ' is the log. decrement which is observed with the open multiplier.

It is assumed throughout that the needle makes small oscillations, and takes no positions in respect to the coil in

which their reciprocal influence is appreciably lessened (compare end of **82**).

Corollary.—The differential equation of the damped oscillating needle is $\frac{d^2x}{dt^2} + \frac{p}{K}\frac{dx}{dt} + \frac{D}{K}x = 0$, where x denotes the angle of deflection at the time t. The integration of the equation gives for the case $p = < 2\sqrt{KD}$, the periodic condition in the form

$$x = C \, . \, e^{-\frac{1}{2}p/Kt} \, . \, sin\,(\sqrt{KD - \tfrac{1}{4}p^2}/Kt)$$

Hence follow the laws 1 to 9.

78A.—Measurement of Electric Currents of Short Duration and of Quantity of Electricity.

If a current flows through a galvanometer for a time short in proportion to the period of oscillation of the needle, it communicates to the needle a velocity, and consequently a (small) deflection proportional to the quantity of electricity (quantity, current-integral, or charge, $\int idt$), which flows through the section of the conductor. Let C be the ordinary reduction-factor of the galvanometer (**64**, **69**). Since the small deflections are observed with mirror and scale, we may substitute for C the reduction-factor c for 1 scale-division, when, A being the distance of the scale, $c = C/2A$. Let τ be the period of oscillation (**52**); a (or e measured in scale-divisions) the deflection. If the needle be undamped, we have—

$$Q = C\tau/\pi \, . \, a \qquad \text{or } Q = c\tau/\pi \, . \, e \qquad (1)$$

Proof.—If x be the deflection of the swinging needle at the time t, $u = dx/dt$, its angular velocity, and if $D = MH(1 + \Theta)$, the directive force, and K the moment of inertia of the needle, the equation of motion $du/dt = -D/K \; sin \; x$ is applicable. Multiplying by $u = dx/dt$, we have $udu = -D/K \, . \, sin \; xdx$, which by integration gives $\frac{1}{2}(u_0{}^2 - u^2) = D/K(1 - cos \; x) = D/K 2 sin^2 \frac{1}{2}x$, u_0 being the velocity for $x = 0$. For the instant of the greatest deflection ($x = a$) we have $u = 0$, and $\frac{1}{2}u_0{}^2 = 2D/K \, . \, sin^2 \frac{1}{2}a$. Remembering that $D/K = \pi^2/\tau^2$, we obtain, exactly as in the case of the pendulum,

$$u_0 = 2\pi/\tau \, . \, sin \; \tfrac{1}{2}a, \text{ and for small } a, \; u_0 = \pi/\tau \, . \, a$$

If q be the dynamic galvanometer constant (**78**), the quantity of electricity Q gives the needle the angular velocity $u_0 = Qq/K$. Since by (**78**), eq. 9, $q/K = MH(1+\Theta)/KC = \pi^2/C\tau^2$, $u = Q\pi^2/C\tau^2$. We have seen that $u_0 = \pi a/\tau$, and the combination of the two equations gives $Q = Ca\tau/\pi$, *Q.E.D.*

Damped Needles.—In this case the proportion between arc of oscillation e_1 and quantity of current still subsists, but the absolute measurement of the latter requires the ratio of damping k to be known (**51**, compare also **78**).

If we call the logarithmic decrement $\Lambda = nat\ log\ k = 2{\cdot}3026$ *Briggs' log k*, then (Table 21)

$$Q = c\tau/\pi \cdot e_1 \cdot k^{1/\pi \,.\, tan^{-1}\,\pi/\Lambda} \tag{2}$$

This follows from **78**, 6. See same for simplified calculation.

The quantity of electricity Q is naturally obtained in the same units in which the reduction-factor C is measured, *e.g.* in [cm. g.] or in ampere seconds (coulombs) = 0·1 [cm. g.] On the charges of Leyden jars see **85**, III.

If the arc of oscillation is so great that the proportionality between arc and scale readings no longer subsists, these must be reduced as in (**49**) to the sine of half the deflection to one side (compare proof above), since the velocity of passing through the position of equilibrium is proportional to this, as in the case of the pendulum. From an observed deflection $= e$ scale-divisions, therefore, $\frac{1}{3}\frac{1}{2}e^3/A^2$ is subtracted, where A is the distance of the scale from the mirror. (Compare also **79**.)

Permanent Deflection.—If the discharge of the quantity of electricity Q through the galvanometer can be rapidly repeated for a period of time at regular intervals (N times per second), the needle takes a permanent deflection a. Then $Q = C \cdot a/N$, or $c \cdot e/N$.

Measurement of Short Periods of Time, Electromotive Forces, or Resistances by Current-Impulses.—An electromotive force E acts through the time t; the product Et is called the "time-integral" or "short integral of *E.M.F.*" If E is not constant, as, for instance, with an induction coil or an earth-inductor, the sum of the products Edt over all the time-elements dt, that is, $\int Edt$, must be substituted for Et.

If w be the resistance of the circuit, the current-strength at each instant is $i = E/w$, and the quantity of electricity passing in time t is

$$Q = \frac{Et}{w} \quad \text{or } Q = \frac{1}{w}\int_0^t Edt$$

Hence the measurement of Q can be employed—

(1) with known E and w, for the measurement of short intervals of time during which the circuit is closed, as, for instance (Pouillet), times of fall and impact, velocity of shot, etc.;

(2) with known Et or $\int Edt$, for the determination of a resistance (**82**, II.);

(3) with known resistance for the determination of integrals of electromotive force Et or $\int Edt$ (**80**).

79.—Methods of Measuring Currents of Short Duration by Multiplication and Recoil.

In measuring currents of short duration with a damped needle (**51**), especially, for instance, in the measurement of induced currents, it is often advantageous to repeat the impulse at regular intervals. In this case, from the damping of the needle, there finally results a constantly maintained movement, exactly like that of a clock pendulum, which at each swing receives an impulse from the driving weight, but by friction and the resistance of the air is so damped that a series of swings maintains a constant amplitude.

Therefore, if this final result be employed, we obtain an observation which can be easily repeated, and from which an exact mean can be taken; and, further, it is not important that the needle should be at rest at the commencement of the experiment.

It is assumed that the oscillations remain so small, or that the damping ring is so broad, that a constant ratio of damping obtains.

I. *Method of Multiplication.*

The proceeding is quite analogous to the example of the clock pendulum already adduced. An impulse is imparted to

the needle, which swings out and turns back. At the instant when it passes its position of equilibrium backwards, a second impulse is imparted in the opposite direction to the first, so that it increases the motion of the needle. At the following passage through the point of equilibrium, another impulse is imparted in the same direction as the first, and so on. The oscillation is each time wider, till it reaches an amplitude at which that given by previous impulses is only just maintained, and of course this limit is the sooner reached the stronger the damping.

Assuming that small oscillations are employed, which are observed by the mirror and scale (**48**), the limiting arc is proportional to the increase of velocity through a single impulse, and is therefore proportional to the quantity of electricity passing through the galvanometer in a current of short duration.

We may calculate the arc of oscillation a, which the needle previously at rest receives from a single impulse without any damping from the limiting angle A obtained by multiplication, the ratio of damping k or the logarithmic decrement $\lambda = log\ k$ is known (**51**). For moderate damping

$$a = \tfrac{1}{2}A(k-1)/\sqrt{k}$$

and, accurately (Table 21B), $a = \dfrac{A}{2}\ \dfrac{k-1}{k} k^{1/\pi \,.\, tan^{-1}\pi/\Lambda}$, where $\Lambda = log\ nat\ k = 2{\cdot}3026\ Briggs'\ log\ k.$

From a the quantity of electricity Q corresponding to a single current-impulse can be reckoned by **78**A, 1.

Proof.—The needle in swinging out passes through its position of equilibrium with the velocity u_0; therefore by **78**, 6

$$u_0 = \frac{\pi}{\tau}\frac{A}{2}k^{1/\pi \,.\, tan^{-1}\pi/\Lambda}$$

On its return it has the velocity $u_1 = u_0/k$. The difference $u_0 - u_1 = u_0(k-1)/k = u$ is the addition produced by the impulse. This alone, without damping, would correspond to the deflection $a = \tau/\pi \,.\, u_0(k-1)/k$. The substitution of the value of u_0 as above gives the expression.

II. *Method of Recoil.*

This method, which is employed with more powerful impulses, yields at the same time the ratio of damping of the needle.

The needle is set in motion by a single impulse, and is allowed to swing out, back, and out in the opposite direction, then, at the instant of again passing its position of equilibrium (scale-division which the needle indicates when at rest), a second impulse is given it in the opposite direction to the first. By this the needle will be thrown back again, since it has lost velocity by the damping. It is now allowed again to turn twice, and again thrown back at the moment when it next passes its position of equilibrium, and so on. When this proceeding has been several times repeated, the throw of the needle takes a constant value, and this takes place the sooner, the stronger the damping. When this is the case the oscillations are of the form graphically represented in the diagram annexed, in which the times are the abscissæ, and the

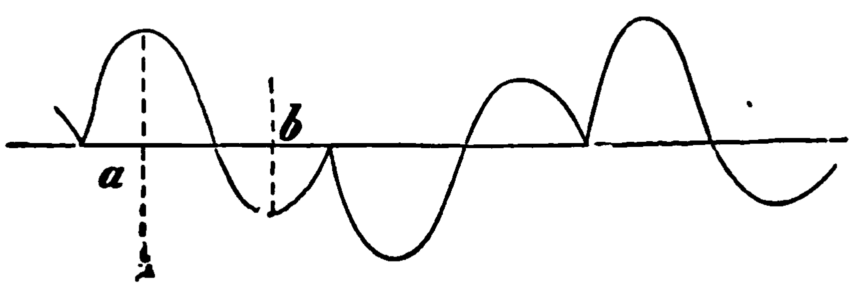


Fig. 83.

scale-divisions, reckoned from the position of equilibrium of the needle, are the ordinates.

The establishment of these regular oscillations will be hastened if the first impulse be enfeebled, and the more so the feebler the damping is. If, indeed, there were no damping, the first impulse should only amount to half the succeeding ones, as follows from the figure.

The method of recoil yields, on taking the mean of the corresponding observations, four turning-points on the scale. The difference a between the two outer we will call the greater arc of oscillation, the difference b of the two inner the smaller arc. (See Fig. 33.)

Then obviously the ratio of damping is $k = a/b$.

The arc of oscillation produced by a single impulse is—

$$\alpha = \tfrac{1}{2}(a^2 + b^2)/\sqrt{ab} \,.\, k^{-1/\pi \,.\, tan^{-1}\Lambda/\pi}$$

Up to $k = 1{\cdot}1$ the exponential factor may be omitted without an error exceeding $\frac{1}{1000}$, and, up to $k = 2$, may be employed in the simpler form $k^{-\Lambda/\pi^2}$. (Compare Table 21B.)

By multiplying α by π/τ we obtain the angular velocity communicated by a single impulse.

Larger deflections are reduced to the sine of the $\frac{1}{2}$ angle (see p. 348).

Sometimes it is advantageous to give the impulse at the end of the third or fourth oscillation instead of as described.

Proof similar to the above.

Cf. W. Weber, Electrodynamische Maassbestimmungen insbesondere Widerstandsmessungen, *Abh. d. K. Sächs. Ges. d. Wiss.* 1, 341, 1846; also J. C. Maxwell, *Treatise on Electricity*, vol. ii. pars. 750, 751. On the influence of duration and right timing of the impulses see Dorn, *Wied. Ann.* xvii. 654, 1882.

80.—The Earth-Inductor (W. Weber).

I. *The Production of known Integrals of E.M.F.*

A coil of wire of the total area of coils f (**83**) is rotated in the magnetic field H; the plane of the coils forming before and after the rotation the angles ϕ_1 and ϕ_2 with the direction of H. Then $\int Edt = H . f(\sin\phi_1 - \sin\phi_2)$, App. 20. ϕ is counted from 0° to 360°. In this way integrals of *E.M.F.* can be produced of any required magnitude. If the plane of the coils be vertical, the horizontal component may be taken as H, and the azimuth with regard to the magnetic meridian as ϕ.

Ordinarily the coils are turned 180° from one east and west position to the other, when

$$\int Edt = 2Hf$$

II. *Measurement of Inclination.*

This measurement rests on a comparison of the currents produced by the horizontal and vertical components in the same revolving coil (inductor).

Since the scale-readings of the galvanometer (reduced when large to the sine of $\frac{1}{2}$ deflection to one side; *cf.* p. 348) are proportional to the current-strengths, and the latter to the required components, the ratio of the scale-readings gives the tangent of the angle of inclination.

The axis of rotation M of the earth-inductor may be placed horizontally or vertically. An "induction-impulse" will be caused by rapidly turning the coil through 180°, the plane of the coil both before and after the rotation being perpendicular to the required component of terrestrial magnetism.

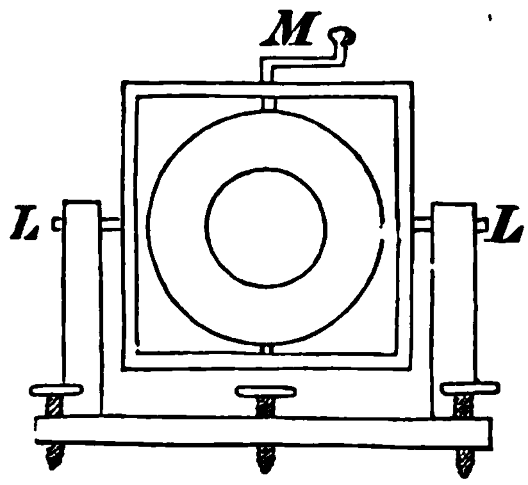


Fig. 84.

To measure the current produced by this rotation, a galvanometer, with a suspended needle, which has a sufficiently long time of oscillation, should be employed. Ordinarily a double astatic needle is used. The narrow coil of the galvanometer serves to damp the needle, and, if these are not sufficient, the damping is increased by a copper casing inserted in the coil.

The coil (or damper) must be so broad that the ratio of damping is the same for both sets of inductions, otherwise errors arise in the method of multiplication, and to a less extent in that of recoil.

Induction by the Vertical Component.—By turning on LL, the coils are placed horizontally, and, by the aid of a magnetic needle, the axis M is brought into the magnetic meridian.

The axis LL must next be carefully levelled by means of the foot-screws and a spirit-level, and its position must not be afterwards changed. By means of the foot-screw at the back (shown in the middle of the figure) the axis of rotation M of the coils is now made exactly horizontal; that is, so that the bubble in a spirit-level placed upon it keeps the same position in the tube when it is turned end for end on the equally thick pins of M. Now a set of induction observations must be made, according to I., previous article, in which, for each impulse, the coil must be turned 180°.

Induction by the Horizontal Component.—The inductor is placed as shown in the figure, with the coils vertical and

against one of the stops, and a spirit-level is placed on the top of the axis *M*, with its tube in the magnetic meridian, and the central foot-screw is turned till the position of the air-bubble is unaffected by turning the coils 180°. When this is the case, the axis *M* is in a vertical plane, perpendicular to the magnetic meridian.

We now make a second set of observations precisely as before.

Method of Induction.—Both sets of induction observations are similarly carried out, preferably by the method of recoil (**79**, II.), because in this variability of damping has a lesser influence. The multiplication is either carried to its constant limit of arc; or, beginning with the needle at rest, the same number of impulses is given in each set of observations, and in each case a like number of arcs of the same order are added together. This sum or the limiting arcs, or, in the method of recoil, the expression $(a^2+b^2)/\sqrt{ab}$, are denoted by S, distinguished in the different positions of the axis by the indices 1 and 2, when the inclination J is given by

$$\tan J = \frac{S_1}{S_2}$$

Testing of the Instrument.—That the two opposite positions given by the two stops really differ by 180° is known by means of a small plane mirror, silvered on both sides, attached to the axis *M*. The eye is brought to the same height as the mirror, and perhaps a meter distant, so that a vertical mark (*e.g.* a window-bar) is visible in it. On rotating to the second position the mark must again appear.

A second test consists in proving that the plane of the coils, when resting against the stops, is perpendicular to the magnetic component to be measured. This may usually be determined with sufficient accuracy for the horizontal component by a compass with right-angled case held against the frame, and for the perpendicular by the spirit-level. In other cases the annexed arrangement (Fig. 85) may be employed, which is fixed to the stops, and limits the rotatory play to perhaps 30°. With this limited angle of rotation a set of induction observations is made on each side,


Fig. 85.

of which the resulting final deflections will be unequal if the position is incorrect.

An error of 1° in the fulfilment of these two conditions only causes a vanishing error in the result, and to provide against it the greatest care must be taken in adjusting *MM* with the spirit-level.

Sources of error are more easily avoided if, instead of working with horizontal and vertical axes, observations are made with the axis nearly in the angle of inclination, and that position exactly determined in which no induction takes place (Schering). The inclination of the axis must then be determined with the theodolite by means of the attached mirror.

Cf. W. Weber, *Abh. d. Gött. Ges. d. Wiss.* Bd. 5, 1853.

81.—Magneto-inductor (Gauss, Weber).

Integral electromotive force of any required magnitude is easily obtained by sliding a magnet bar in a bobbin. By changing between two fixed positions, integral values are obtained in the two directions of equal value but opposite sign. In order to have the values very constant, positions are chosen such that in their neighbourhood the induction is zero.

Absolute Integral Values.—The sliding of a magnet of magnetism M into the middle of a long narrow bobbin of n windings per unit of length, gives the value $\int Edt = 4\pi n . M$ (App. 20).

Double Magneto-Inductor.—The arrangement shown in the figure is specially suitable for constant induction-impulses. The double magnet is pushed quite through the bobbin, the end positions being regulated by adjustible stops (pieces of felt, etc.), so that near them a small movement produces no electromotive force. Care must be taken that the movement of the magnet does not influence the needle of the galvanometer directly.

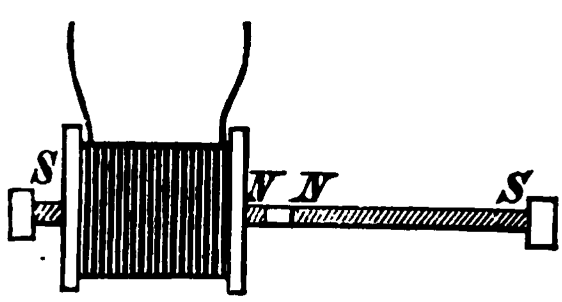


Fig. 86.

Measurement of Resistance.—If the coil is closed through a galvanometer, the quantity of current is inversely proportional to the total resistance. The quantity may be measured by **78A** or **79**.

w is the resistance of inductor and galvanometer; a is the deflection with this resistance, and a_1 with an added resistance w_1. Then $w_1/w = (a - a_1)/a_1$. Here w may be expressed in terms of w_1, or *vice versâ*. If w_2 be substituted for w_1, the deflection a_2 is produced. Then $w_1/w_2 = (a - a_1)/(a - a_2) \cdot a_2/a_1$.

a is to be understood strictly in the sense of **78**, 5, or **79**, I. or II. If the damping is nearly constant, see also **80**, methods by induction.

Induction - impulses with this inductor are also very convenient and effective for comparisons of resistance with the differential galvanometer or the bridge (**71**A and B) if the resistances give no strong extra currents. The advantage is obtained that the magnitudes of the oscillations may be employed for interpolation. (*Pogg. Ann.* cxlii. 418.)

81A.—Coefficient of Magnetic Induction.

The magnetism of a bar is altered temporarily by the action of a feeble magnetising force (say, not exceeding 4 [$cm.^{-\frac{1}{2}} g.^{\frac{1}{2}} sec.^{-1}$] or twenty times the horizontal component of the earth's magnetism) in proportion to the force. The coefficients of strengthening and weakening of permanent magnets are nearly equal; and in ordinary magnets the change of magnetism for unit magnetising force amounts to about 1·5 to 2 cm. g. units per $cm.^3$ of steel, or 0·2 to 0·3 per grm. The amount depends on the form, hardness, and chemical character of the steel, and is somewhat greater for unmagnetised than for magnetised material.

A magnet hanging north and south has therefore a somewhat greater magnetism by some hundredths of a cm. g. unit per grm. of steel than one lying east and west. The excess in proportion to the magnetism of the bar itself is called in measurements of terrestrial magnetism the induction-coefficient by the horizontal component (Lamont).

For the measurement of this excess Weber employs a

narrow extended bobbin, which can be rotated 180°, and which must be longer than the magnet bar. The rotation is from one north and south position to the other, while the coil of the bobbin is closed through a galvanometer with slowly swinging needle. The deflection a_0 is observed when the bobbin alone is turned, a when it is turned with the bar in its axis, and a_1 when a small bar of known magnetism M_1 (**62**) is rapidly brought from some distance and inserted to its middle in the empty bobbin.

The magnetism induced in the first magnet by the north and south position is then $m = \frac{1}{2}M_1(a + a_0)/a_1$; the magnetism induced by unit magnetic field is m/H where H is the horizontal component of the earth's magnetism (**59**, Table 22); and, lastly, the induction-coefficient $\Delta = m/M$ where M is the total magnetism of the bar.

For these observations multiplication will be generally employed (**79**). When the damping is feeble, time may be saved by taking for a in each case an arc of oscillation of like number of impulses, or better, the sum of an equal number of arcs of oscillations of the same order, instead of inducing to the limiting arc.

Observation with a Current.—Instead of reversing the position of the bobbin and magnet with regard to the earth's magnetism, it may remain unchanged if they are surrounded with a second coil wound round or slid over the first, in which a measured current i can be closed and opened, or rapidly commuted. The inner coil then receives an electromotive force from the magnet, and another from the outer coil. The observations are exactly similar to those above described. The magnetic field in the current coil is $= 4\pi ni$, if n be the number of windings in the unit of length.

Cf. F. K., *Wied. Ann.* xxii. 417, 1884; Sack, *ibid.* xxix. 53, 1886.

81B.—Determination of a Strong Magnetic Field.

I. *By Induction* (Verdet).

A small plane conductor (wire hoop) of the area of coil f, with its plane perpendicular to the lines of force, is suddenly

brought into the field from a great distance, or withdrawn from it. It is connected with a mirror galvanometer of not too small period of oscillation.

If H be the strength of the field, an electromotive force of the integral value fH will be induced. Turning through 180° instead of withdrawal will produce $2fH$.

If the galvanometer needle makes a first deflection e (measured in scale-divisions), then

$$H = P \cdot e/f$$

Determination of the Experimental Constant P.

1. *With the Earth-Inductor* (**80**).—An earth-inductor of area of coils f_0 is permanently included in the same circuit, and when turned through 180°, gives the deflection e_0; H_0 being the intensity of terrestrial magnetism perpendicular to the plane of the inductor coil (**59**). Then (Quincke, *Wied. Ann.* xxiv. 349, 1885)

$$P = 2H_0 f_0 / e_0$$

2. *With the Magneto-Inductor.*—A long bobbin with N windings per unit of length of its axis is permanently connected in circuit with the galvanometer and small inductor. A short magnet of moment M (**62**) is rapidly pushed into, or withdrawn from, the middle of the bobbin. Let the galvanometer needle make a first deflection e'. Then (end of **63**, and App. 20)

$$P = 4\pi NM/e'$$

3. *From the Reduction-Factor of the Galvanometer.*—Let the ordinary reduction-factor for cm. g. sec. be $= C$ (**64**, II.; **69**), or the reduction-factor C for 1 scale-division be $C = /2A$, where A is the scale-distance (**66**). If, further, k be the ratio of damping, $\Lambda = nat\ log\ k$ (**51**), τ the time of oscillation of the undamped needle, and w the resistance of the galvanometer and small inductor in absolute measure, *i.e.* for cm. g. the resistance in ohms multiplied by 10^9 (App. 21); then

$$P = Cw\tau/\pi \cdot k^{1/\pi \cdot \tan^{-1}\pi/\Lambda}$$

On the calculation of the exponential factor see Table 21B, and the remarks on **78**, 6.

Proof.—The *E.M.F.* integral is by measurement Hf (App. 20); in the first determination of P, $2H_0f_0$ (**80**, I.); in the second, $4\pi NM$ (App. 20). Since the resistance is the same, it immediately follows that

$$P = Hf/e = 2H_0f_0/e_0 = 4\pi NM/e'$$

The expression under 3 is derived from the fact that on the one hand the quantity of electricity of the impulse $Q = Hf/w$, and on the other, by **78**A, 2,

$$QC\tau/\pi \cdot e \cdot k^{1/\pi \cdot \tan^{-1}\pi/\Lambda}$$

II. *With a Bifilar Galvanometer*, see **77**B, II.

III. *From the Change of Resistance of Bismuth.*

The resistance of bismuth increases in the magnetic field (Righi): for small H more rapidly, but from about $H = 10000$ (cm. g.) with approximate uniformity, till at $H = 20000$ to 23000 it reaches double its original value. A spiral of compressed bismuth wire experiences the greatest change when placed across the lines of magnetic force.

The method of measurement is obvious when we know the resistance to be a function of the magnetic field. The table or curve must be obtained empirically. For spirals of pure pressed bismuth wire, Lenard gives the mean resistance w in the magnetic field H as

$H=0$	2000	4000	6000	8000	10000	12000	14000	16000 cm. g.
$w=1$	1·049	1·126	1·217	1·316	1·420	1·527	1·634	1·740

Lenard, *Wied. Ann.* xxxix. 619, 1890.

81c.—Distribution of Magnetism in a Magnet.

A short narrow bobbin of N coils is brought from a great distance over a determinate section of a magnet, or, more conveniently, is removed from it. The integral of *E.M.F.* thus induced in the coil (**78**A, App. 20) is equal to the product of $4\pi N$ and the magnetic moment m of the unit of length of the bar at the place from which the coil is drawn off (or the number of lines of force which pass through this place).

If the coil be drawn rapidly off while connected through a galvanometer, the quantity Q of an impulse may be measured by **78A**. If w is the total resistance of the circuit, Qw is the integral of *E.M.F.*, and

$$m = Qw/(4\pi N)$$

The sum of m for all the units of length, or the integral $\int m dx$ over the entire bar, is the magnetic moment of the bar.

82.—Absolute Measurement of Resistance (Weber).

Compare **78-80**, and App. 19-21.

I. *From the Damping of an Oscillating Magnet.*

Let—

K be the ratio of damping of a magnetic needle in closed galvanometer (**51**);

$\Lambda = log.\ nat.\ k$, the natural logarithmic decrement;

Λ' the same with open circuit (air-damping);

τ the period of oscillation of the undamped needle;

G the statical galvanometer constant, *i.e.* the ratio of the (small) deflection to the current-strength in unit magnetic field, with torsionless suspension (p. 346);

M the magnetism of the needle;

Θ its coefficient of torsion;

H horizontal intensity of terrestrial magnetism.

1. The absolute resistance in Weber's electromagnetic units is then—

$$w = \frac{\pi}{2\tau}\frac{G^2}{\Lambda - \Lambda'}\frac{M}{H(1+\Theta)}\sqrt{1 + \frac{\Lambda^2}{\pi^2}} \qquad (1)$$

On the determination of M/H see **59**, II.

Constant of Sensitiveness.—For a circular multiplier of n windings of radius R with a short needle in the centre, $G = 2\pi n/R$ (**64**, II.) The breadth b, and the thickness h of the coils, and the polar separation $\mathfrak{l}$ of the needle, if small compared to R, may be taken into calculation by multiplying G by

$$1 - \tfrac{1}{8}b^2/R^2 + \tfrac{1}{12}h^2/R^2 + \tfrac{3}{16}\mathfrak{l}^2/R^2$$

For a close coil with a long needle, G must be determined

empirically by a current which is passed entire through a tangent compass, and with a shunt through the galvanometer (Dorn). If the angles of deflection are ϕ and ϕ', and the torsion-coefficient Θ and Θ' respectively, while G' is the constant of the tangent compass, and v the shunt-factor (**64**, III.), then

$$G = vG' . \tan \phi / \tan \phi' (1+\Theta)/(1+\Theta')$$

II. *With the Earth-Inductor.*

Let an earth-inductor with vertical axis of rotation (**80**) be closed through the galvanometer. Retaining the signs as above, and adding

f the area of coils of the inductor (**83**);
a the deflection of the needle by a single induction-impulse without damping, in the sense of **78A**, and turning on its vertical axis as in **80**.

2. If the constant of sensitiveness of the galvanometer is known, or determined as above, the ratio of damping is only required so far as is needed for the calculation of a; and

$$w = \frac{2\pi}{1+\Theta} \frac{fG}{a\tau} \tag{2}$$

3. Instead of the constant of sensitiveness an exact knowledge of the damping will suffice. K being the moment of inertia of the needle,

$$w = \frac{8}{\pi} \frac{f^2 H^2 \tau}{a^2 K} \frac{\Lambda - \Lambda'}{\sqrt{\pi^2 + \Lambda^2}} \tag{3}$$

4. K may be eliminated by aid of the known equation (App. 10), $K = MH(1+\Theta)\tau^2/\pi^2$, yielding

$$w = \frac{8\pi}{1+\Theta} \frac{f^2}{a^2 \tau} \frac{H}{M} \frac{\Lambda - \Lambda'}{\sqrt{\pi^2 + \Lambda^2}} \tag{4}$$

The quantity a may be determined in 2 by multiplication or recoil, but in 3 and 4, in order to obtain the damping at the same time, the method of recoil must be used. If the

two stationary arcs of oscillation in absolute measure $=a$ and b, we must write (**79**, II.)

$$\alpha=\tfrac{1}{2}\frac{a^2+b^2}{\sqrt{ab}}\left(\frac{b}{a}\right)^{1/\pi\,.\,tan^{-1}\Lambda/\pi}$$

and

$$\Lambda=2{\cdot}3026\ (log\ a-log\ b)$$

On simplification of calculation see Table 21B and pp. 345, 351.

The above methods are derived from **78**, from eqs. 7 and 8 of which

$$\frac{M^2G^2}{w}=2K\frac{\Lambda-\Lambda'}{T}\qquad\text{or }=2K\frac{\Lambda-\Lambda'}{\tau\sqrt{1+\Lambda^2/\pi^2}}$$

whence

$$w=\tfrac{1}{2}\frac{M^2\tau}{K}\frac{G^2}{\Lambda-\Lambda'}\sqrt{1+\Lambda^2/\pi^2}$$

Substituting $MH(1+\Theta)\tau^2/\pi^2$ for K we obtain eq. 1.

An induction-impulse from the horizontal component H gives further the current-quantity $2fH/w$, and thus imparts to the needle an angular velocity (eq. 7):

$$u_0=\frac{2fH}{w}\frac{q}{K}=\frac{2fH}{wK}\sqrt{2wK\frac{\Lambda-\Lambda'}{T}}=\frac{fH}{\sqrt{w}}\sqrt{\frac{8(\Lambda-\Lambda')}{KT}}$$

Hence follows $w=f^2H^2/u_0{}^2\,.\,8(\Lambda-\Lambda')/KT.$ If we further write (**78**, eqs. 6 and 3) $u_0=\pi/\tau\,.\,\alpha$ and $T=\tau\sqrt{1+\Lambda^2/\pi^2}$, we obtain eq. 3.

Eq. 2 is obtained from eq. 3 by inserting from **78**, eqs. 7 and 3,

$$\Lambda-\Lambda'=q^2T/2wK=G^2M^2\tau\sqrt{\pi^2+\Lambda^2}/2wK\pi$$

and then instead of K^2 writing $M^2H^2(1+\Theta)^2\tau^4/\pi^4$.

All quantities are to be expressed in cm. g. sec. w divided by 10^9, then gives the resistance in ohms.

Methods 2 and 3 allow of the use of an astatic needle.

On inconstant ratio of damping comp. K. Schering, *Wied. Ann.* ix. 471, 1880. Also the self-induction of the coil involves a correction, on which see Dorn, *Wied. Ann.* xvii. 783, 1882. Local variations of terrestrial magnetism may also demand correction.

III. *From the Reciprocal Induction of two Conductors* (Kirchhoff).

Let the reciprocal induction-coefficient of two coils $=P$. This is calculated from the form and position of the coils, and

is in general very complicated. A simple case is that of a long bobbin of radius r, uniformly wound with s windings per unit of length, and encircled by a narrow short coil of m windings (Roiti, Himstedt). Neglecting a correction dependent on the limited length of the first bobbin, P then $= 4\pi^2 r^2 sm$.

Let the current i arise or vanish in the primary bobbin, inducing an *E.M.F.* integral $\int Edt = Pi$.

The current-quantity induced in the secondary circuit is then $Q = Pi/w$, which, measured by **78A**, gives w in absolute measure.

By aid of a contact breaker in the primary circuit, by which the current i is interrupted N times per sec. (**37A**), and which is provided with a disjunctor which allows only the opening or closing current to pass, the measurement of Q may be made by permanent deflection (Roiti, Himstedt).

The deflection of a galvanometer in the secondary circuit being thus α_1, while the inducing current gives with the same galvanometer α_2, we have

$$w = NP \,.\, \tan \alpha_2 / \tan \alpha_1$$

For $NPi/w = C \tan \alpha_1$ and $i = C \tan \alpha_2$. For arrangement and corrections see Himstedt, *Wied. Ann.* xxvi. 547, 1885.

IV. *From the Heat produced.*

Let A current i cm.$^{\frac{1}{2}}$g.$^{\frac{1}{2}}$sec.$^{-1}$ (**64**) evolve in a conductor in t sec. the quantity of heat q (**29** to **31**); then the absolute resistance w of the conductor is—

$$w = A \,.\, \frac{q}{i^2 t} \frac{\text{cm.}}{\text{sec.}}$$

A is the absolute mechanical equivalent of the unit of heat, *e.g.* $A = 42{,}000{,}000$ *cm.*^{2}*g.sec.*^{2}/water-gram calories (App. 7).

83.—Determination of the Area of the Coils of a Bobbin of Wire.

I. *From the Measured Diameters.*—A direct (but either troublesome or inexact) method is that of measuring the diameter of each layer of wire in many places (with the katheto-

meter or calipers), or the circumference (with tape measure). Of course, the thickness of the wire must be deducted from the diameter measured to the outer surface of the layer.

If the number of windings is N, and the inner and outer diameters r_0 and r_1 with uniform winding, the area—

$$f = \tfrac{1}{3}\pi N(r_0^2 + r_0 r_1 + r_1^2)$$

II. *From the Length of Wire.*—If the wire is not too thin the sum of the coil-areas of a bobbin may be measured while winding it by determining the number of coils and the length of the wire wound on.

If the coils are circular and form a layer of rectangular section; calling—

l = the total length of wire;
n = the number of coils;
h = the depth of the layer of wire—(the breadth is of no consequence);

the effective area of the coils when acting on a distant object is—

$$f = \frac{l^2}{4\pi n} + \tfrac{1}{12}\pi n h^2$$

From the sinking in of the wire and the compression of the covering, the value obtained in this way will be more or less too large.

Compare H. Weber, *Der Rotationsinductor*, Leipzig, 1882.

III. *By Magnetic Effect* (F. K.).—Let the same current traverse the bobbin and a mirror tangent galvanometer with a coil of radius R, and a short needle which is acted on by both parts of the current.

The axis of the bobbin lies east and west. Its centre has the distance a from the needle, which is placed alternately either east and west of it (first position), or north and south (second position), **59**, II.

Let the deflection of needle be ϕ when both parts of the current act in the same direction, and ϕ' when that in the tangent compass alone is commuted.

The required area of coils is then for the first position

$$f=\frac{a^3\pi}{R}\frac{\tan\phi+\tan\phi'}{\tan\phi-\tan\phi'}$$

and in the second position

$$f=\frac{2a^3\pi}{R}\frac{\tan\phi+\tan\phi'}{\tan\phi-\tan\phi'}$$

For since the moment of rotation of the needle, caused by the current i in the coil and the tangent compass together must balance that of terrestrial magnetism H, we have (for the first position) $2Mif/a^3 . \cos\phi + Mi2\pi/R . \cos\phi = MH\sin\phi$, or

$$2i(f/a^3+\pi/R)=H\tan\phi$$

and similarly $\qquad 2i(f/a^3-\pi/R)=H\tan\phi'$

which by division yield the above expression.

Corrections.—1. The pole-separation $\mathfrak{l}$ of the needle is taken into account if $R(1-\frac{3}{16}\mathfrak{l}^2/R^2)$ be substituted for R.

2. The diminution of force in proportion $1/a^3$ is not rigidly accurate. Let L be the length and r_1 and r_0 the inner and outer radii of the bobbin, while a is so large that L^4 and R^4 may be neglected in comparison. Let $(r_1^5-r_0^5)/(r_1^3-r_0^3)$ be called k. Then the above expressions for f are divided, in the first position by $1+1/a^2 . (\frac{1}{2}L^2-\frac{9}{10}k)$, and in the second by $1+1/a^2 . (\frac{2}{4}\frac{7}{0}k-\frac{3}{8}L^2)$.

Distance of Coils is measured by placing the tangent compass successively on opposite sides of the bobbin and taking $\frac{1}{2}$ the distance between the two positions of the suspending fibre for a.

Variations of current are the less important, the smaller ϕ' is. If ϕ' is on the opposite side to ϕ, it must be taken as negative.

Compare F. K., *Wied. Ann.* xviii. 513, 1883.

83A.—Self-Induction of a Conductor (Maxwell).

The coefficient of self-induction (or electromagnetic capacity or potential of a conductor on itself) is the factor by which the velocity of change di/dt of the current in the conductor

must be multiplied, in order to obtain the *E.M.F.* of induction (of the extra current). The measurement may be made by the bridge.

1. *After Dorn.*—The conductor to be measured is inserted in the branch a. G is a mirror galvanometer of considerable period of oscillation, and resistance γ. A second galvanometer is placed in the undivided current. The resistances are so arranged that there is no current in G, when the galvanometer in the undivided current shows the deflection ϕ. The current is now interrupted, and the extra current caused in a gives the needle of G the (momentary) deflection e. Its time of oscillation is τ and its ratio of damping k (**51**) $\Lambda = log.\ nat.\ k.$ $S = [\gamma(a+b+c+d) + (a+b)(c+d)]/d.$

Then the coefficient of self-induction Π of the conductor a is

$$\Pi = S \cdot \frac{\tau}{\pi}\frac{C'}{C}\frac{e}{\phi}k^{1/\pi \cdot \tan^{-1}\pi/\Lambda}$$

C and C' are the reduction-factors of the principal and bridge galvanometers respectively (**64**, II., **66**, **68**A, **69**). If the former is a tangent-compass, $tan\ \phi$ must be substituted for ϕ.

Proof.—If $J = C\phi$, the principal current, and i_1 the current in a, then in the first case we have $i = J(b+d)/(a+b+c+d)$. If i_1 vanishes, the *E.M.F.* in a for the time t is $\Pi di_1/dt$, the current i in G (*v.s.*) is $i = \Pi\dfrac{di_1}{dt}\dfrac{c+d}{\gamma(a+b+c+d)+(a+b)(c+d)}$. Substituting J for i, and recollecting further that $ad = bc$ because there is no current in the bridge; and therefore $(b+d)(c+d) = (a+b+c+d)d$, we find that $i = \Pi dJ/dt \cdot 1/S$. Therefore

$$\int idt = \Pi J/S, \text{ or } \Pi = S\int idt \cdot 1/J = S \cdot C'\,\tau/\pi \cdot e \cdot k^{1/\pi \tan^{-1}\pi/\Lambda} \cdot 1/C\phi$$

C' and C being the reduction-factors of the two galvanometers.

The difference of sensitiveness of the two galvanometers would be too great to allow of direct comparison (see **69** on methods of comparison of C'/C by aid of resistances, shunts, etc.).

The resistances being expressed in [cm./sec.], we obtain Π in absolute measure, *i.e.* in [cm.]; if the resistances are measured in ohms, Π is obtained in ohm-secs. or earth-quadrants (App. 20).

On calculation see p. 345 and Table 21B.

2. *After Rayleigh.*—Instead of measuring the principal current J, it is simpler to measure on G itself the permanent deflection e' caused by the insertion of a small resistance w in the branch a. Then—

$$\Pi = w \,.\, \tau/\pi \,.\, e/e' \,.\, k^{1/\pi \,.\, tan^{-1}\pi/\Lambda}$$

For, after the insertion of w, a current arises in the bridge

$$C'e' = J\frac{wd}{(a+b)(c+d)+\gamma(a+b+c+d)} = J\frac{w}{S}$$

3. *Comparison of two Self-Induction Coefficients* (Maxwell).—The conductors with self-induction coefficients Π and Π' are inserted in the branches a and c, together with rheostat resistances, while b and d are free from induction. The resistances are adjusted so that the needle of G is unaffected either by the permanent current, or by opening or closing the circuit. Then

$$\Pi/\Pi' = b/d$$

This relation follows from No. 1 (p. 366), for we can conceive the deflection 0 as the sum of two opposite deflections arising from Π and Π', viz. $\alpha = A \,.\, \Pi/S$ and $\alpha = A\, \Pi'/S'$, where α stands for the expression $C'/C \,.\, \tau/\pi \,.\, 1/\phi \,.\, k^{1/\pi \,.\, tan^{-1}\pi/\Lambda}$., which is common to both. S and S', however, only differ by their numerators d and b. Thus $\Pi/\Pi' = S/S' = b/d$.

4. *Comparison of a Self-Induction Coefficient with the Capacity of a Condenser* (Maxwell).—The bobbin, with the self-induction coefficient Π, is placed in the branch a; while a condenser of capacity C in electromagnetic measure (**86**) is connected on parallel with the branch d, *i.e.* the ends of d are connected by short wires with the coatings. If the needle of G remains at rest, both when the current is flowing and at opening and closing, $\pi/C = a \,.\, d = b \,.\, c$.

Compare also—1. Dorn, *Wied. Ann.* xvii. 783, 1882; 2. Lord Raleigh, *Phil. Trans.* 1882, ii. 661; 3. Maxwell, *Electric.* ii. Art. 757; 4. *ibid.* Art. 778. On determination of self-induction and capacity by alternating currents with the telephone, M. Wien, *Wied. Ann.* xliv. 689, 1891; on methods with the magneto-inductor, F. K., *ibid.* xxxi. 594, 1887.

ELECTROSTATICS.

84.—On Electrostatic Work in general.

Insulation.—Shellac furnishes good insulating supports. Surface-conduction of glass is prevented by washing and drying in dust-free air, and it is permanently lessened by coating with shellac by aid of heat. Paraffin is a good insulator, but easily put out of shape. Quartz plates cut parallel to the axis are serviceable at high temperatures.

Induction machines are most easily kept dry by aid of small petroleum lamps. Care is required with regard to fibres of the pulley-cord and similar interferences.

Protecting Covers.—In order to prevent inductive action from and on surroundings, enclose apparatus and conductors in metal casings connected to earth (cardboard lined with tinfoil, wire netting, or strips of tinfoil pasted on, or, better, inside the glass-covers, etc.)

Precautions in Measuring with Separated Quantities of Electricity.—To avoid frictional electricity, mercury in commutators is contained in thimbles placed on an insulating support.

Measurements of small capacities require conductors, commutators, and such-like also of small capacity. Commutators, for instance, may be made of fine platinum wire on sticks of shellac, and the connections may be made by arches of platinum wire held in the same way.

Condensers for exact measurements must retain no residual charge. Air-condensers, or those insulated with paraffin, are to be preferred; in making the latter, the plates are completely immersed in melted paraffin; impurities, such as oil, being carefully avoided (see Arons, *Wied. Ann.* xxxv. 291, 1888.

Carriers free from residual charge may be made from small metal tubes filled not quite to the ends with paraffin, in the axes of which the conducting wires are carried.

Production of Constant Potentials.—Best by batteries of many cells, *e.g.* Spamer's chromic acid elements (p. 269). Higher potentials are given by Leyden batteries.

Zero of Potential.—Differences of potential only are measurable. To obtain in all the apparatus of an experimental arrangement a common point of commencement, and fix a potential "zero," all the conductors which belong to it are connected to earth (water-pipes, gas-pipes, earth-plates).

In the following sections potential is always understood to mean potential-difference from the chosen zero.

84A.—Comparison of Electrostatic Potentials.

I. *With the Sine Electrometer* (R. Kolrausch).

The force is measured with which a magnetic needle is repelled by a horizontal arm which can be rotated, together with the casing of the instrument, about a vertical axis over a graduated circle.

If the needle is deflected through the angle ϕ, while the repelling arm is retained at a position with regard to it which has been once for all determined, the potential to which the needle and arm are charged is

$$V = C\sqrt{\sin\phi}$$

The constant position is known by the coincidence of a mark on the needle-mirror with the image of one on the casing of the instrument reflected in it. Both are seen through a slit, reflected in a second mirror attached to the case. ϕ is the angle through which the instrument must be turned from zero to bring this about.

According to the strength of the potentials to be determined the observations may be made either with different needles or with different angles between these and the repelling arm, adjusted by rotating the case of the instrument in its base plate, which permit of very different values of C. These must be made comparable with each other by experiment. For this purpose, the same Leyden battery of large capacity is successively connected with the instrument under the conditions to be compared and ϕ observed. The loss of electricity during the intervals of observation is eliminated by alternated observations at equal short intervals.

In order that no loss may occur by the production of a residual charge in the battery, it is advisable that the battery should have been already kept some time charged (compare R. Kohlrausch, *Pogg. Ann.* vol. lxxxviii. p. 497, 1853).

II. *Quadrant-Electrometer* (Thomson).

Setting up.—The two pairs of quadrants must act equally on the "needle." All parts being connected to earth, the instrument is adjusted so that the centre of symmetry of the needle is brought over a diameter separating the quadrants.

Exact Adjustment.—The needle is turned by means of the suspension so that with quadrants to earth the deflection on electrifying the needle to a high potential will be a minimum, or, better, that in changing to an equally high potential of opposite sign, the opposite deflections shall be equal.

After a fresh strong charge of the needle, the equilibrium is sometimes unsteady. Mere waiting is often sufficient, but otherwise, if practicable, a quadrant may be moved, or a change in the position of the needle may be tried; or, lastly, a weaker charge may be given.

If the quadrant pairs and the needle are charged to the potentials Q_1, Q_2, and N respectively, the needle is deflected to the angle α (measured with mirror and scale, **48**, **49**). If C be the electrometer-constant (**84**C)

$$\alpha = C(Q_1 - Q_2)\left(N - \frac{Q_1 + Q_2}{2}\right)$$

With Auxiliary Charge. Heterostatic Arrangement.

1. *Quadrant Connection.*—One of the quadrant pairs is kept permanently at zero potential, and the needle at a potential, high compared with that to be measured, either by means of a battery of many cells, of which the other terminal is put to earth, or, where great constancy is not necessary, by a Zamboni pile or a Leyden jar, which often forms part of the instrument.

If the second quadrant pair, which had also been put to earth, be now charged to potential V, the needle will be deflected to an extent nearly proportional to V. The usual

inequality of deflection to the two sides of the position of rest is eliminated by commutation of V and measurement of the double deflection.

The deflections for $+V$ and $-V$ differ, according to the formula, by $\pm V/2N$ of the entire deflection; *e.g.* about 4 per cent if the needle is charged with 50 chromic acid elements (100 volts), and 2 volts are measured.

2. *Needle Connection.* — The two pairs of quadrants are permanently charged to opposite and equal potential by connection with the poles of a battery of many cells, of which the middle is put to earth. If the needle, previously brought to zero potential by connecting to earth, be now raised to the potential V which is to be measured, a deflection is produced (see above formula) which is proportional to V. The sign of V may then be changed by a commutator as above. Absolute equality of the opposite potentials of the quadrants is not important.

In observations with auxiliary charge, the sensitiveness must be determined from time to time (**84**c, I.) because of variation in the auxiliary potentials.

Without Auxiliary Charge (Idiostatic Arrangement) for Larger Potentials.

The needle and one quadrant pair are put to earth. The second quadrant pair, previously put to earth, is raised to the potential V, and gives a deflection e. Then

$$V = 2C\sqrt{e}$$

Instead of putting to earth the quadrants connected with the needle, these and the needle may be charged, and the opposite pair put to earth.

Large scale-deflections must be reduced to arc (**49**); but the correction may be combined with that for calibration of the instrument (**84**c). In exact measurements the square root of the total deflection, caused by the simultaneous changes of the quadrants and the potential sign, must be used in calculation, which eliminates the contact potential differences of the different materials

used in construction of the instrument. Neglect of this precaution may under some conditions lead to important errors. (See Hallwachs, *Wied. Ann.* xxix. 1, 1886.)

Constancy is dependent solely on that of the suspension in respect of directive force, length, and torsion, and can be secured in great measure by the use of very fine metallic wires.

The sensitiveness of the quadrant electrometer varies with vertical changes of position of the needle. In many cases this may be usefully employed to vary the sensitiveness, but for exact measurements it should be as nearly as possible in the middle of the quadrant casings, the position of minimum sensitiveness. Stretching of the suspension has then less influence, and the symmetry of the deflections is least disturbed by curvatures of the needle, unequal heights of the quadrants, etc.

To damp oscillations a vane in a liquid (sulphuric acid free from dust) is generally employed, which often causes disturbances of zero and other inconveniences. The little vane must be hung by a very fine platinum wire (0·1 - 0·05 mm.), which passes through the surface of the liquid at the centre of rotation. On avoidance of liquid damping, see Hallwachs, *loc. cit.* pp. 19-28. For forms of quadrant electrometers see Kirchhoff, Brauly, Mascart, Edelmann, Hallwachs.

III. *Capillary Electrometer* (Lippmann).

A very finely drawn out glass tube contains mercury and 60 per cent sulphuric acid in contact with each other. A potential difference between them causes a change of capillary pressure at the point of contact, and hence a displacement, which for small potential difference is proportional to the latter. Either the displacement may be observed with the microscope, or the change of pressure required to restore the point of contact to zero.

For larger electromotive forces, which, however, must not exceed 2 volts, a table of displacements may be empirically constructed. The contact to the sulphuric acid is made by means of mercury. Care must be taken to ensure good wetting of the surfaces by motion before an observation.

IV. *Other Electrometers.*

1. *Hankel's Electrometer.*—In this instrument a gold or aluminium leaf, and a pair of lateral charged plates, play the same parts as the needle and pairs of quadrants of II.; and the same modes of connection are employed. A microscope with ocular micrometer measures the displacement of a fine tooth on the leaf. For sharp focussing, this must be lighted with a small gas-flame at some distance.

The instrument has very small capacity and instantaneous reading. At its maximum sensitiveness it will indicate to 0·01 volt. The sensitiveness may be varied by approach or separation of the lateral plates, or alteration of the auxiliary charge. With idiostatic connection (*i.e.* without auxiliary charge) it will measure to about 100 volts.

2. *Leaf Electroscopes.*—Aluminium or gold-leaf electroscopes with suitable graduated arcs may be employed to measure potentials of from 50 to 1000 volts. The gold leaves should be surrounded as far as possible by metal casings connected with earth (**84**). The scale must be empirically graduated.

On an electroscope for potentials of 50 to 200 volts see Exner, *Wiener Berichte*, 95, II., 1088, 1887; for a similar one with a strip of aluminium turning on an axis, for 500 to 10,000 volts, Braun, *Wied. Ann.* xxxi. 857, 1887, and xliv. 771, 1891.

3. *Righi's Reflecting Electrometer*, for potentials of 3000 to 25,000 volts, is specially suited as an auxiliary to the absolute electrometer (**84**B). This instrument is a modified quadrant electrometer. Its deflections e, measured with mirror and scale, are nearly proportional to the potentials V. Empirical graduation is necessary (**84**C); which may be based on the formula $V = c\sqrt{e(1 - c'e)}$ as a simplification. On liquid damping see end of II.

Comparison of Electromotive Forces.—The *E.M.F.* of a battery is proportional to the potential difference of its poles, and hence to the deflections of the electrometer.

The potential difference between different points of a closed circuit, *e.g.* the terminal *P.D.* of a dynamo, may also be determined by electrometry.

Comparison of Resistances.—The resistances to be compared are arranged in series in the same circuit, and the two ends of one are connected with the opposite terminals of the electrometer, and its deflections observed. If similar observations are made with the others, the measured potential differences are proportional to the resistances. The constancy of the current during the measurements must be tested.

To determine *electrolytic resistances* in this way, the ends of the electrolyte, which is contained in a calibrated narrow glass tube, are connected with the electrometer by electrodes without polarisation (zinc in vessels with zinc-sulphate solution which are suitably united to the ends of the tubes).

Bouty, *Ann. der Ch.* (6), iii. 433, 1884.

Measurement of Current-Energy.—An electrometer in idiostatic connection is united with one end of a resistance w in the circuit, while the other end of w is put to earth. The deflection, divided by w, is proportional to current-energy /sec. in the length w if the latter contain no electromotive force. This is true also of alternating currents.

84B.—Absolute Measurement of an Electrostatic Potential (W. Thomson).

A movable plane circular plate of area f, surrounded with a guard-ring, stands at the distance a above a larger fixed plate. The potential difference $V - V_0$ then causes an attractive force $k = \frac{f}{8\pi}\left(\frac{V - V_0}{a}\right)^2$. If this force and also f and a are measured, $V - V_0$, or where $V_0 = 0$, V itself may be calculated.

It is more accurate to substitute for f

$$f = \frac{\pi}{2}(R^2 - R'^2) + \frac{R'^2 - R^2}{1 + a/(0{\cdot}22b)}$$

where R and R' are the radii of the movable disc and of the aperture of the guard-ring respectively, and $b = R' - R$, the breadth of the narrow separating space.

To avoid the difficulty of exact measurement of a, the

following method may also be adopted. The fixed plate is permanently electrified to a constant and sufficiently large potential of the opposite sign to that to be measured, while the movable plate is brought to zero by putting to earth. Let the attractive force in this case be k. The movable plate is now brought to the potential V which is to be measured. In order to bring the plate to its normal position, the distance a must now be increased by l, when

$$V = l\sqrt{8\pi k/f}$$

Kirchhoff's Balance.—The movable plate forms the scale of a balance which is in the same plane as its guard-ring when the index of the balance is at 0. A stop on the opposite scale prevents its closer approach to the fixed plate below it. If the latter is charged to V, p grams must be placed in the opposite scale to make the balance begin to tip. The potential is calculated by the formula given above, substituting $k = g\,.\,p$. The instant of tipping is determined by the breaking of a galvanic current which flows through the stop into the second scale.

For details see Quincke, *Wied. Ann.* xix. 561, 1883; Czermak, *Wiener Berichte*, xcvii. 307, 1888.

The electrostatic unit adopted here is $=300$ volts (App. 20).

Compare Maxwell, *Electricity*, i. §§ 217, 218; Wiedemann, *Electricität*, 3rd ed. i. p. 175.

Striking Distance, Length of Spark, is sometimes a convenient means of estimating absolute potentials (see Table 27B).

84c.—Adjustment and Calibration of Electrometers.

The deflections are observed which are produced by known potentials applied to the electrometer.

I. *Instruments for Small Potentials.*

Known potentials are given by normal elements of known *E.M.F.* (p. 269). The deflections produced by different elements, singly and together, are used in calibration.

Potentials of more exactly known value, and more easily controlled as to constancy, may be obtained by connecting the electrometer with different points of a rheostat of high resistance, traversed by a current, and of which one terminal is put to earth. The entire terminal potential difference of the rheostat must be as large as the greatest potential which is required for calibration. The current-strength i is measured in amperes. If a resistance of w ohms is contained between the electrometer connection and the terminal to earth, the potential at the electrometer is iw volts.

II. *Instruments for High Potential.*

1. *With an Absolute Electrometer.*—For potentials of about 1000 volts and over. The potentials are produced by an induction machine connected with a Leyden battery. It is best first to adjust the absolute instrument to the required potential, and then to charge the two instruments to one somewhat higher. By approaching a point connected with earth, touching with a handkerchief, or some similar device, the potential is very gradually reduced, and as soon as the absolute instrument indicates that required, the instrument which is being tested is read off.

2. *With Galvanic Battery.*—For potentials to some thousand volts. Even when the potential of the battery is insufficient, testing may be carried out in the following ways.

(*a*) The battery is connected with one terminal of a potential-multiplier, and the electrometer with the other. If z be the ratio of increase of potential, and E the *E.M.F.* of the battery, zE is the potential of the electrometer.

Hallwachs, *Wied. Ann.* xxix. 300, 1886; Exner, *loc. cit.* **84A.**

(*b*) The pole P_1 of the *insulated* battery is permanently connected with the inner coating of a large Leyden battery, of which the outer coating is put to earth. P_1 is first put to earth while the electrometer to be tested, or another sensitive auxiliary electrometer, is connected with pole P_2, and its deflection n_1 noted, which corresponds to V, the potential of the battery. The electrometer is then disconnected from P_2

and connected with P_1, which is disconnected from earth, and raised by an electric machine, or other source of potential, to the potential V, as shown by the electrometer being again deflected to n_1. The pole P_2 is now at a potential of $2V$. Proceeding in an analogous way any required multiples of V may be obtained, and communicated to the electrometer. It is essential that the Leyden battery should be of very large capacity as compared with the electrometer. Compare F. Braun, *loc. cit.* **84**A.

85.—Quantity of Electricity in a Leyden Jar.

I. *With the Electrometer.*

Since the quantity of electricity in a given condenser is proportional to the potential, different charges of the same battery may be compared by means of the electrometer (**84**A). The "residual charge," *i.e.* that quantity of electricity which remains in a jar after a momentary discharge, has no effect on the potential. The indications of the electrometer are therefore proportional to the "available" charge, *i.e.* to that quantity which is discharged by a momentary connecting of the two coatings.

II. *With Lane's Unit Jar.*

In charging the battery, the quantity of electricity added may be determined by carefully insulating the battery, connecting one coating with the electrical machine and the other with a unit jar. Every spark discharge of the unit jar corresponds to a definite increase of the charge in the coating of the battery connected with it. The residual charge is included in the measurement.

The indications of the unit jar, when the striking distance is varied, are reduced to each other experimentally by comparing with the sine electrometer the potentials to which the different striking distances correspond. If the striking distances are not too small, they may be assumed to be approximately proportional to the potentials. Compare also Table 27B.

III. *With the Galvanometer.*

The quantity of electricity in a battery may be determined by its discharge through a galvanometer with sufficiently insulated winding, as explained in (**78A**). The danger of the discharge striking through the insulation of the wire or changing the magnetism of the needle is diminished by including in the circuit a great resistance, such as a wet thread.

The unit quantity of electricity, as measured in electromagnetic measure, 1 [*cm. g.*] is 10 *amp. secs.*, or 3×10^{10} electrostatic [*cm. g.*] units. (Compare App. No. 11 and **19A**.)

IV. *With the Air Thermometer* (Riess).

The depression of the column of fluid by an electrical discharge traversing the wire is proportional to the product of the quantity of electricity discharged and its potential before the discharge. It is here assumed that the resistance of the wire in the thermometer bulb is very great compared with that of the remainder of the circuit through which the discharge is effected.

In this way quantities discharged from the same Leyden jar or battery may be compared by the air thermometer, for since the charge is here proportional to the potential, the quantities discharged vary as the square roots of the depressions produced in the air thermometer.

86.—Electrical Capacity.

The capacity *c* of a conductor is that quantity of electricity which it holds when it is charged to the unit potential while the conductors within its sphere of induction are kept at zero potential. Capacity depends not only on the form of the conductor, but also on its position with regard to surrounding objects (observer, table, wall, etc.)

Condensers.—Capacity here means simply that of one (the inner) coating of the "collector." For purposes of accurate measurement, air-condensers (R. Kohlrausch), or sometimes

those with paraffin insulation (**84**) are employed. Those insulated with glass, mica, or oiled silk do not exactly follow the simple laws of condensers on account of residual charge and surface conduction. Their capacity usually rises with increasing temperature up to over 1 per cent per 1°.

I. *From the Dimensions (in Electrostatic Units).*

The capacity of condensers of simple form may be calculated. a is the constant distance apart of the coatings, and the formulæ suppose air as the dielectric. When this is not the case, the results must be multiplied by the specific inductive capacity (**86A**).

Parallel Surfaces.—At a distance a relatively very small to the surface f, approximately $c = f/(4\pi a)$.

Circular Plate Condenser of radius r. Approximately $c = \frac{1}{4}r^2/a$ (since $f = r^2\pi$). More accurately, d being the thickness of the plates, by multiplication by the factor

$$1 + \frac{1}{r\pi}\left[a + a \,.\, log\ nat\ \frac{16\pi r(a+d)}{a^2} + d \,.\, log\ nat\ \frac{a+d}{d}\right]$$

Guard-Ring Condenser.—Let R be the radius of the collector-plate, R' the inner radius of the guard-ring, and $b = R' - R$, the width of the narrow separating space between them. Then approximately $c = (R + R')^2/(16a)$. More accurately by multiplying the expression by the factor

$$1 - 8a(\beta\ tan\ \beta + log\ nat\ cos\ \beta)/(R + R')$$

where $\beta = tan^{-1}\frac{1}{2}b/a$. The formula assumes b to be small as compared to the thickness of the plates.

Kirchhoff, *Abhandl.* p. 113; for another formula see Maxwell, *Electricity*, i. § 201.

Cylindrical Condenser of length l, inner radius R and outer $R + a$. If a be small compared to r and l, approximately $c = \frac{1}{2}l \,.\, R/a$.

Isolated sphere of radius r. $c = r$.

An electrostatic capacity measured in cm., divided by 900,000, gives the capacity in microfarads (App. 20A).

II. *With the Electrometer.*

The remarks in **84** and at the beginning of **86** must be carefully noted, especially with small capacities. The conductors which are being compared must be so placed that they do not influence each other by induction.

The oscillations of the electrometer may be rapidly brought to rest by suitable shunting in and out of elements in the earth connection of the proper part of the electrometer.

1. *By Division of Charge.*—The conductor I., connected with the electrometer (of capacity γ), is charged to the potential V. The conductor II., which has previously been put to earth, is now connected, and the potential sinks to V'. Then

$$c_2 : (c_1 + \gamma) = V - V' : V'$$

The method is especially suitable for large capacities on which γ has little influence. It makes considerable demands on the insulation.

The capacity γ of an electrometer may be similarly compared by division of charge with a condenser.

Capacities, especially of quadrant electrometers and of connections, are apt to be underrated. The change of γ with the deflection must also in some cases be taken into consideration.

2. *By Neutralisation.*—A galvanic battery of many elements is closed through a large (rheostat) resistance R, and the conductors I. and II. (of which the capacity is to be compared) are connected respectively with the two ends of R. An earth connection is made with a suitable point in R, so chosen that if the conductors are disconnected from R and connected together, their (opposite) charges should exactly neutralise each other. When this is the case their capacities are inversely as the corresponding sides of R. In this case the capacity of the electrometer which is used to prove neutrality is of no consequence.

For proof see **63**, I., p. 267.

The method may be modified in several ways. The conductors may, for instance, be connected with the two poles of the *open* battery, and earth connection made to such a point in the latter that the cells are divided in the ratio $c_1 : c_2$.

Also in comparing nearly equal capacities, cells closed through a resistance may be added to one pole of the open battery, and by connecting one conductor with a suitable point in this resistance the potentials may be brought to such a relation that the charges exactly neutralise. On the carrying out of the method see Lebedew, *Wied. Ann.* xliv. 289, 1891.

3. *In the Wheatstone Bridge.*—If the earth-contact is so adjusted that on making contact at a with an electrometer previously put to earth it remains undeflected, then

$$c_1 : c_2 = w_2 : w_1$$

A battery of many cells and a large resistance must be used.

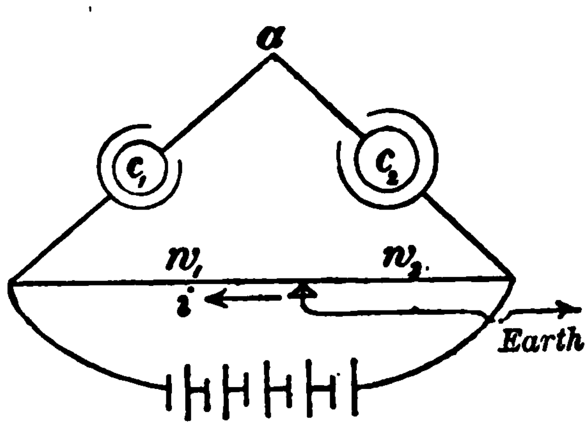


Fig. 87.

Proof.—The potentials of the outer coatings are $-iw_1$ and iw_2. If p be the common potential of c_1 and c_2, their charges are $(p+iw_1)c$ and $(p-iw_2)c_2$. Since on account of the insulation their sum must $=0$, it follows that $p(c_1+c_2)=i(w_2c_2-w_1c_1)$, and hence for $p=0$, $w_1c_1=w_2c_2$.

Methods 2 and 3 are suitable for the calibration of sets of standard capacities.

III. *With the Ballistic Galvanometer.*

These methods are only satisfactory for condensers of large capacity.

1. *By Single Charge.*—The two condensers to be compared are charged to the same potential, and discharged singly through the same galvanometer (**78**A). The capacities are proportionate to the deflections. A galvanic battery (p. 270) gives equal potentials, so that the multiplication method (**79**) may be employed with advantage. Leyden jars may also be charged to the same potential by the electric machine if connected during charging, or by connection with an electrometer.

2. *By Neutralisation.*—The charges are of opposite signs as in II. 2. Equality of charge is tested by the galvanometer, through which both condensers are discharged at once.

3. *In the Wheatstone's Bridge* (Sauty).—If w and w' are so regulated that on commuting the total current no deflection is produced, $c : c' = w' : w$. If for the battery and galvanometer we substitute induction coil and telephone, the method becomes applicable to small capacities. Care must be taken that the resistances are free from self-induction (Palaz).

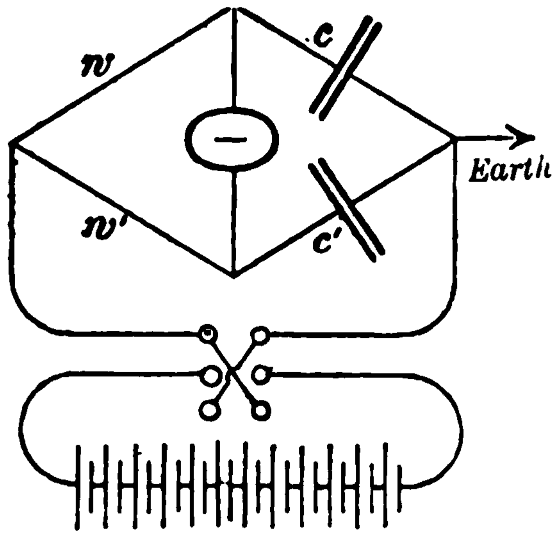


Fig. 88.

4. *Absolute Capacity.*—Discharge through a galvanometer (compare No. 1) gives the capacity c in electromagnetic measure (App. 20A), if the electromotive force E of the charging battery be known. The quantity Q of electricity is calculated by **78A**, when $c = Q/E$.

5. *According to Maxwell.*—E and the galvanometer constant are eliminated by the following method:—The battery is closed through the galvanometer and a resistance R, which must be very large compared to that of the battery. A constant deflection $= e_0$ is produced. A condenser charged by the same battery gives on discharge through the galvanometer the deflection e. The resistance of the battery and galvanometer $= W$, the period of oscillation of the undamped needle $= \tau$, the ratio of damping $= k$ and $\Lambda = nat\ log\ k$ (**51**). Then

$$c = \frac{\tau}{\pi} \frac{1}{R + W} \frac{e}{e_0} \cdot k^{1/\pi \cdot tan^{-1} \pi/\Lambda}$$

(Table 21B). If the deflections are large, e is corrected to double the sine of the half-angle, and e_0 to the tangent (**49**, Table 21A).

If a sufficiently large resistance R is not at disposal, the elements may be arranged in parallel groups of equal numbers, when the above expression must be divided by the number of the groups. A shunt to the galvanometer may also be employed; compare **64**, III.

τ in secs., R and W in ohms, and in 4, Q in ampere-secs., and E in volts give the capacity in farads (App. 20A). 1 microfarad $= 10^{-6}$ farads $= 10^{-15}$ electromagnetic $= 9 \cdot 10^5$ electrostatic cm. g. units.

IV. *With the Galvanometer* (after Siemens).

1. *Comparison.*—The condenser is connected alternately with the battery and the galvanometer by a rapidly vibrating commutator (tuning-fork contact-breaker). The permanent deflection of the needle (or its tangent, **49**) is, *cet. par.*, proportional to the capacity. The smaller the latter, the faster may the contact breaker vibrate without fear of incomplete charge or discharge. The capacity of the conducting wires must, however, be independently determined and allowed for.

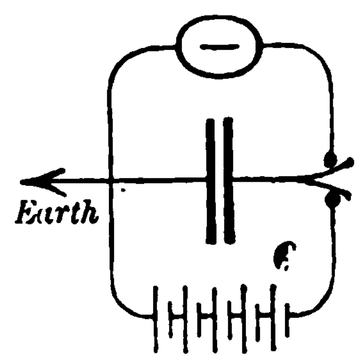


Fig. 89.

2. *Absolute Determination.*—From the known reduction-factor c of the galvanometer (**64**, **69**), the electromotive force E of the battery, and the rate of vibration N of the commutator (**37A**), the capacity is obtained in absolute measure, since obviously cEN is equal to the mean current-strength which is obtained from the deflection e as ce. It is, however, simpler in this case also to close the battery through the galvanometer and a great resistance R. Adopting the same notation as in III. 5, we have

$$c = \frac{1}{N} \frac{1}{R + W} \cdot \frac{e}{e_0}$$

It is advantageous to take $\frac{e}{e_0}$ not far from 1, in which case the influence of correction to the tangents is inconsiderable. On methods with insufficiently great R, see III. 5.

3. *Zero Methods.* — The continuous series of discharge currents may be passed through one coil of a differential galvanometer, while through the other is passed a constant current from the same battery regulated by intercalated resistance. The condenser may also be inserted in a branch of the Wheatstone bridge. The derived circuit in which the condenser and commutator are inserted acts like a resistance of $1/Nc$.

Siemens, *Pogg. Ann.* cii. 66, 1857. For details of these methods and of the tuning-fork contact-breaker see Klemencic, *Wien. Berichte*, lxxxix. 298, 1884; Himstedt, *Wied. Ann.* xxix. 560, 1886; and xxxiii. 1, 1888.

86A.—SPECIFIC INDUCTIVE CAPACITY (Faraday).

I. *With the Condenser.*

The capacity of a condenser is, *ceteris paribus*, proportional to the specific inductive capacity or "dielectric constant" K of the intermediate insulating layer or "dielectric." K is taken as unity for air, but the capacity of gases is also frequently referred to vacuo.

A specific inductive capacity may thus be measured by the ratio of two capacities (**86**); but the choice of methods is limited by the conductivity of the dielectric; and the errors so produced are those which must be principally considered. These are generally diminished by rapid alternation of charge and discharge.

Plate or cylindrical condensers are employed; the latter best fulfil the theoretical requirement that the whole of the lines of force should pass through the dielectric. The capacity of the conducting wires, etc., must be deducted in each measurement.

COMPLETE INSULATORS.—By the methods in **86**.

Fluids.—It is advantageous to compare the experimental condenser which contains air or the fluid with a second constant condenser of similar capacity. The method of Siemens (**86**, IV. 1), and especially that with alternating currents and telephone (**86**, III. 3), are suitable for the measurement.

Gases require a zero method since their S.I.C. is little different from 1; *e.g.* with the electrometer and neutralisation (**86**, II. 2), or with the galvanometer and contact-breaker (**86**, IV. 3).

For liquids see among others Silow, *Pogg. Ann.* clviii. 306, 1876; Palaz, *J. de Phys.* (2), v. 370, 1885. On gases and vapours, Klemencic, *Wien. Bericht.* xci. 712, 1885; Lebedew, *loc. cit.*

Solid Bodies are interposed between the plates of a condenser in sufficiently broad plane-parallel discs. If a be the separation of the condenser plates (small as compared to their

radius), c_0 the capacity with air alone, and c that with the plate of the dielectric of thickness d (**18**) interposed, then

$$1/K = 1 - a(1 - c_0/c)d$$

Proof.—$c_0 = \frac{1}{4}r^2/a$ (**86**, I.) The disc of dielectric acts like a layer of air of the thickness d/K. In addition to this there is still a layer of air of the thickness $a - d$, and therefore $c = \frac{1}{4}r^2/(a - d + d/K)$. r is eliminated by division; see Boltzmann, *Pogg. Ann.* cli. 482 and 531, 1874.

Separation of Plates.—The measurement of a and the correction for the connections is avoided if the condenser-plates can be separated parallel to each other in a measurable degree. After interposing the solid plate, an increase of separation of e is needed to restore the original capacity. Then (compare above proof)—

$$K = d/(d - e)$$

The method is also applicable to liquids, which are poured into a plane-parallel trough between the plates.

A zero method is most convenient to prove equality of capacity. A system of five equidistant condenser-plates (Gordon, *Phil. Trans.* 1879, 417) may be used for this purpose. One of the external plates (No. 1) is movable, and together with the other (No. 5), is connected to earth. No. 3 is connected with the needle of an electrometer, and with one pole of an induction apparatus, of which the other is put to earth. The connections of the electrometer correspond to those of the electrodynamometer in the Wheatstone Bridge (Fig. 70, p. 318). When No. 1 is so adjusted that the needle is unaffected, the capacities have the ratio (1, 2) : (2, 3) = (4, 5) : (3, 4). The adjustment is made with and without the dielectric plate, the difference of position being e. The greater K is, the more important are errors in the adjustment.

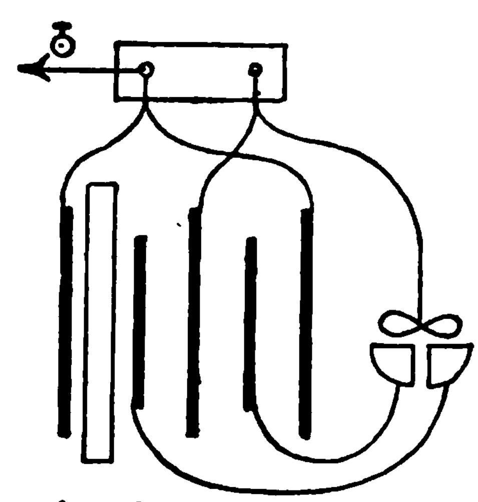


Fig. 90.

A telephone may be substituted for the electrometer (Winkelmann).

On employment of three plates only see Winkelmann, *Wied. Ann.* xxxviii. 161, 1889; but compare also Cohn, *ibid.* xlvi. 1892. On employment of the differential inductor, Elsas, *ibid.* xliv. 654, 1891.

Incomplete Insulators.—The results of the above methods may be materially vitiated by mere traces of conduction. The two following methods give correct results with bodies with a conductivity up to about $5 \times 10^{-13} Hg$.

(1.) If a condenser c discharges itself through a coil of coefficient of self-induction Π (**83A**), electrical oscillations are produced between the condenser-plates of the period $\tau = \pi\sqrt{c\Pi}$, and τ is therefore, *cet. par.*, proportional to $\sqrt{c}$. On the production, measurement, and employment of oscillations for the determination of c and K see Schiller, *Pogg. Ann.* clii. 535, 1874.

(2.) The method of Cohn and Arons, *Wied. Ann.* xxviii. 454, 1886, depends on the measurement of the rate of charge of condensers.

II. *By Measurement of Force* (Attraction or Repulsion).

The reciprocal force exerted by two conductors kept at constant potential is proportional to the specific inductive capacity of the medium in which they are placed.

The deflections produced by a constant potential (Daniell's batteries, accumulators) in a suitably constructed quadrant electrometer in idiostatic connection (**84A**, II.) are observed when it is filled with air and with the fluid to be examined (Silow, *Pogg. Ann.* clvi. 389, 1875). The deflections, corrected to proportionality with the square of the potential difference (**84C**), are proportionate to the dielectric constant or specific inductive capacity of the medium. The needle is suspended by a fine wire, which serves at the same time as a conductor.

Traces of conduction cause interference by polarisation; and on this account it is advisable to charge with alternating currents (**72**; induction coil, rotating commutator), and by this means even such bodies as alcohol, water, and solutions up to $k = 10^{-9}$ may be measured (Cohn and Arons). The variability of potential is eliminated by an ordinary electrometer connected in parallel, and read at the same time.

Convection currents from variations of temperature, evaporation, etc., must be carefully avoided.

Cohn and Arons, *Wied. Ann.* xxxiii. 13, 1888; Tereschin, *ibid.* xxxvi. 792, 1889.

III. *From the Velocity of Translation of Electrical Waves.*

According to Maxwell the dielectric constant is equal to the index of refraction for infinitely long waves. The optical extrapolation of the latter has proved in general unreliable; but Hertz's electric waves of meter length fulfil the necessary conditions. The measurement of their velocity in various media offers, therefore, a further method of determining the dielectric constant.

Arons and Rubens, *Wied. Ann.* xlii. 581, 1891; xliv. 206, 1891; Cohn, *ibid.* xlv. 370, 1892.

Dielectric Constants.

(The agreement of different observers is in many cases very unsatisfactory.)

Ordinary glass	$K=$ about 6	Paraffin	2·0
Optical glasses	5 to 10	Water	84
Different micas	6·6 and 8	Alcohol, 95 per cent	27
Sulphur	3·9	Castor oil	4·6
Shellac	3 to 3·7	Carbon disulphide	2·6
Ebonite	2·2 to 2·8	Benzol	2·34
Vulcanised caoutchouc	2·8	Turpentine	2·2
Ordinary ,,	2·2	Petroleum	2·1

86B.—Measurement of very great Resistances.

Special methods are often necessary for the measurements of extremely great resistances, such as those of the insulation of cables. It is to be observed in practice that when the large resistances are also of considerable capacity, the powerful batteries employed must be very constant (Daniell, p. 270), or the feeble currents to be observed will be disturbed by those of charge and discharge.

1. If sufficiently sensitive galvanometers, powerful batteries, and large comparison resistances are at hand, the methods **70**

to **71B** may be employed; and especially the bridge arrangement (**71B**, I.) will serve for resistances up to 10,000,000 if we have a rheostat up to 10,000, and take the ratio of the branch circuits 1 : 1000.

2. If E is the *E.M.F.* of a battery in volts, C the reduction factor of the deflections of a mirror galvanometer to am., then the deflection e of the galvanometer shows the total resistance of the circuit $W = 1/e \,.\, E/C$ ohm.

3. One element closed through a mirror galvanometer of resistance γ and a rheostat resistance R gives a deflection e'; k elements through the same galvanometer, and the resistance w give e; then, assuming that the elements are similar, and that their resistance may be neglected,

$$w = (\gamma + R)k \,.\, e'/e - \gamma$$

Frequently γ may also be neglected.

4. The use of a shunt to the galvanometer gives more exact results than 2 and 3. A battery closed through w, the resistance to be measured and a galvanometer of resistance γ without shunt gives the deflection e. The same battery with the known large resistance R, and the galvanometer in a derived circuit of resistance z gives e'. If w_0 is the resistance of the battery, we have accurately

$$w = e'/e \,.\, [(R + w_0)(z + \gamma)/z + \gamma] - \gamma - w_0$$

With very large resistances γ and w_0 may mostly be neglected as compared to R and w. Then, simply

$$w = e'/e \,.\, R(z + \gamma)/z$$

5. *With the Condenser* (Siemens).—Resistances of so-called non-conductors, *e.g.* the different sorts of gutta-percha and the like, are sometimes too large for galvanometric methods. In these cases, the time of charge or discharge of a condenser may be used. If the potential (**84A**) of a condenser of capacity c (**86**) sinks in the time t from the value V_1 to V_2, the resistance of the path of discharge is

$$W = \frac{1}{c} \frac{t}{\log V_1 - \log V_2}$$

If in this way the value W is found when the condenser stands alone, and W_1 when the two coatings are connected through the resistance w to be measured, the value of w alone (**63**, p. 268) is

$$w = WW'/(W - W')$$

If c is given in absolute measure (farads), the resistance is obtained in similar measure (ohms) by the use of natural logs. $(= 2{\cdot}303$ Briggs' log.) It is indifferent in what measure V is determined.

Proof.—The potential V corresponds to the quantity of charge $Q = cV$. In the time-infinitesimal dt there is lost of this $dtV/W = -dQ = -cdV$. By integration we obtain the above expression.

Conversely we may determine the capacity of a condenser by discharge through known W (Siemens and Halske).

For refinements of the method and formulæ for the rate of the discharge see Klemencic, *Wien. Ber.* xciii. 470, 1886.

DETERMINATIONS OF TIME AND PLACE.

87.—Some Astronomical Terms.

(1.) In the determination of the *place of a heavenly body* the following definitions are made use of:—

Azimuth A: The arc of the horizon between the south of the horizon and the vertical circle of the star.

Altitude h: The arc of the vertical circle between the horizon and the star.

Hour or Declination Circles: Great circles through the pole of the heavens.

Hour Angle τ: The arc of the celestial equator between the south point of the meridian and the hour circle of the star.

Declination δ: The arc of the hour circle between the equator and the star.

Altitude of the Pole φ: The geographical latitude of the place.

Parallactic Angle: The angle between the hour circle and the vertical circle of the star.

These angles have, among others, the following relations:—

$$\sin\delta = \sin\phi\sin h - \cos\phi\cos h \,.\, \cos A \qquad (1)$$

$$\sin h = \sin\phi\sin\delta + \cos\phi\cos\delta \,.\, \cos\tau \qquad (2)$$

$$\cos h\sin A = \cos\delta\sin\tau \qquad (3)$$

$$\cos h\cos A = -\cos\phi\sin\delta + \sin\phi\cos\delta \,.\, \cos\tau \qquad (4)$$

$$\sin\tau\cot A = -\cos\phi\tan\delta + \sin\phi\cos\tau \qquad (5)$$

Vernal Equinox: The ascending node of the ecliptic.

Right Ascension of a Star α: The arc of the equator between the vernal equinox and the hour circle of the star. The equator is divided into 24 h. or 360°. The R.A. is reckoned in a direction contrary to the diurnal motion.

The other arcs of the equator or horizon are reckoned in the same direction as the diurnal motion.

For the places of some of the principal stars see Table 35.

(2.) In determinations of time the following terms are used:—

Sidereal Time z: The arc of the celestial equator between the south point of the meridian and the vernal equinox, the entire equator being reckoned equal to 24 hours.

Sidereal Day: The time between two consecutive culminations of a fixed star. 1 mean day $= 1{\cdot}002738$ sidereal days $= 1$ sidereal day $+ 235{\cdot}9$ mean seconds.

The sidereal day begins with the passage over the meridian of the vernal equinox. A heavenly body therefore passes the meridian (culminates) at the instant when its right ascension is equal to the sidereal time.

In general terms, sidereal time = hour angle + right ascension of a star, or $\tau = z - a$.

True or Apparent Noon: The time of the passage of the sun's centre across the meridian.

True Solar Time: The hour angle of the sun.

Equation of Time: The mean or civil time minus the true solar time.

The astronomical solar day begins at noon, is reckoned from 0 h. to 24 h., and bears the date of the civil day in which it begins.

Since the introduction of "unit time," which is related to a given meridian of geographical east longitude l_0 (15° for Middle Europe, 0° for England, Greenwich), the mean time of a place of east longitude l = unit time $+ 4\ (l - l_0)$ min.

For declination of the sun, sidereal time, and equation of time, see Table 31.

More extended tables in *Nautischen Jahrbuch*, the Berlin *Astr. Jahrbuch*, or the *Nautical Almanac;* also in Bremiker's *Logarithms.* Further or more exact methods, A. Brünnow, *Sphär. Astronomie;* Jordan, *Zeit. und Ortbestimmung;* Wislicenus, *Geogr. Ortsbestimmungen*, Leipzic, 1891, etc.

88.—Theodolite.

For the measurement of angles of azimuth and altitude, one axis of the instrument must be vertical, the other hori-

zontal, and the optical axis of the telescope must be perpendicular to the horizontal axis.

To be independent of the possible eccentricity of the divided circles, readings should always be taken by two Verniers 180° apart. It is most convenient to take the number of degrees from Vernier I. and use the mean of the two readings for the subdivisions only.

I. *Adjustment of the Vertical Axis.*—The axis is vertical when the bubble in the level does not change its position on rotation round this axis. This is most conveniently attained by first placing the level parallel to the line joining two of the foot-screws, and by the use of these bringing the bubble to the centre. The instrument is then turned through 180°, and if any change has been produced in the position of the bubble *half the difference* is corrected by the aid of the foot-screws. Finally, a rotation of 90° is given to the instrument, and by the aid of the third foot-screw the previous position of the bubble is produced. If this process has left any error the first time it must be repeated.

It will, of course, be understood that if necessary the level must be first corrected, so that when horizontal the bubble is exactly in the middle.

II. *Adjustment of the Horizontal Axis.*—(*a.*) The ordinary method assumes that the two pivots of the telescope axis are of equal size. This is tested by levelling the axis and then reversing the telescope (changing the positions of the pivots), and again placing the level in its previous position: the same position of the bubble shows the equality of the pivots.

This being assumed, the axis is known to be horizontal when the level gives the same reading as before, when reversed end for end.

The roundness of the pivots of the telescope axis are tested by turning them while the level is standing on them.

(*b.*) The horizontal position of the axis is tested independently of the equality of the pivots by hanging a long plumb-line at a distance from the theodolite, and observing it at various heights.

(*c.*) Finally, the two theodolite axes are known to be perpendicular to each other as follows:—First, two rather distant

objects are found lying one immediately above the other, which are observed in the telescope by simply rotating it on the horizontal axis. Then the instrument is turned through 180°, and the two former objects observed again. If they both again come into sight by a simple rotation round the horizontal axis, the two axes are perpendicular to each other. A previous adjustment with the level is here unnecessary, but the absence of collimation error (compare III.) is assumed.

III. *Test whether the Optical Axis of the Telescope is Perpendicular to the Horizontal Axis* (*Collimation Error*).—(*a.*) The instrument is directed towards an object lying nearly in the horizon, and then rotated through exactly 180° round the vertical axis; the telescope is then again brought to its former position by rotating it on the horizontal axis. If the same object is again exactly on the cross-wires it shows that there is no collimation error. If there is any difference, *half* of it must be corrected by moving the cross-wires, and the test then be repeated.

(*b.*) Or after directing the telescope to an object as before, the instrument remains fixed, and the telescope is reversed on its supports, and again directed to the same object, which must still appear on the cross-wires.

The equality of the telescope pivots is here assumed.

IV. *Measurement of an Absolute Altitude. Determination of the Horizon and Zenith Points of a Theodolite.*—(*a.*) The instrument is adjusted as in I. to III. The telescope is then directed to the object, and the vertical circle is read off; the instrument is then turned through 180° on the vertical axis, the telescope brought round and again directed to the object, and the vertical circle is again read. The difference of the two readings (attention being paid to their signs) gives twice the zenith distance of the object, the half difference, therefore, subtracted from 90° furnishes the altitude.

The arithmetical mean of the two readings gives the zenith point of the vertical circle, and by adding 90° to this the horizontal point is obtained.

(*b.*) *Artificial Horizon.*—Instead of rotating the instrument, an artificial horizon (mercury bath) may be placed in front of the telescope, and we then obtain the altitude of the object by

measuring the angle between it and its image in the mercury. The zenith and horizon points of the instrument are, of course, readily obtained by this method.

The artificial horizon may also be used in measuring altitudes with the reflecting sextant.

This method is directly available for heavenly bodies at their time of culmination. At other times the altitude is for the mean of the times of the two observations if they are made quickly following one another.

The observation of objects of high altitude is made possible or more easy by placing over the eye-piece a small total reflecting (*e.g.* right-angled) prism. The cross-wires may be illuminated by placing at an angle in front of the objective some sq. mm. of white paper illuminated from one side.

Angle between two Objects.—From the altitudes h and h' of two objects and the difference A of their azimuth, their angular distance w may be found by the equation

$$\cos w = \sin h \sin h' + \cos h \cos h' . \cos A.$$

Repeating Theodolite.—To increase the accuracy of measurements of azimuth, a second vertical axis is provided, concentric with the first, and on which the whole instrument may be rotated, when employed in the following manner:—After adjusting on the second object, the whole instrument with the circle is turned so as to bring the first object again on the cross-wires, and the telescope alone is then turned again to the second, and this is repeated as often as desired. If the telescope has been turned n times, the total angle read on the circle is divided by n.

The total angle is the difference between the first and last reading + as many times 4 right angles (360°) as the index has passed 0°.

89.—Determination of the Meridian of a Place.

I. *From Observations of the Greatest Elongation of a Star.*—The meridian of a place is most simply determined by observations on a circumpolar star, preferably the pole-star itself, at the time when it attains its greatest easterly or westerly

elongation. Since at this time the movement of the star is vertical, it is easy to make the adjustment conveniently and accurately.

If the star is observed at both elongations, east and west, the meridian bisects the angle between the two vertical circles; but since the declination δ of the star and the altitude of the pole ϕ are known (Table 35, *Nautical Almanac*, etc.) an observation on one side only suffices. For the vertical circle of the greatest elongation makes with the meridian the angle θ, which is found by the formula—

$$\sin\theta = \cos\delta / \cos\phi.$$

For the meridian, the vertical circle of the star and its hour circle form at the time of the greatest elongation a right-angled triangle with the hypothenuse $90-\phi$, one side $90-\delta$, and the angle θ opposite this latter side.

The nearer the star is to the pole the more suitable the star is for these observations. The pole-star's greatest elongations are approximately at 7 h. 13 m. and 19 h. 26 m. sidereal time (Table 31).

II. An observation of the *pole-star at any known time* is often sufficiently accurate. From the mean time is calculated the sidereal time z (Table 31), and from this and the right ascension a of the pole-star (Table 35) the hour angle $\tau = z - a$ of the latter, and, lastly, its azimuth A by **87,** 5; or approximately $\theta = (90-\delta) \sin\tau/\cos\phi$.

III. *From Equal Altitudes.*—The telescope of a theodolite of which the axis has been made vertical (**88,** I.) is directed to a heavenly body, and the horizontal circle is read off. Without touching the position of the vertical circle the same body is again observed after its culmination, the telescope being so placed that the body passes over the intersection of the cross-wires. The meridian of the place bisects the two readings of the horizontal circle. The instrument need not have a vertical circle. It is well, for the sake of accuracy, that the change of altitude should be as quick as possible at the time of observation, and therefore the star must not be too near the meridian.

When the sun is observed the vertical wire is placed in

the morning on the one edge, in the afternoon on the other, while the horizontal wire touches, say, the upper edge. The bisecting line of the two positions, however, will not in general pass exactly through the meridian on account of the variation of the sun's declination between the observations, but must be subject to a correction, which may amount to some minutes.

Let t be half the difference of time in hours between the two observations; the hour angle of the sun in degrees is therefore $= 15t$. Further, let ϵ be the variation in the sun's declination during a day (see Table 31 and Bremiker's *Five-Figure Logarithms*, p. 131) and $\epsilon t/24$ the amount of this change in the half time. Then, if ϕ be the altitude of the pole, $\frac{1}{\cos\phi}\frac{\epsilon t}{24}\frac{1}{\sin 15t}$ is the required correction. Of course the observed bisecting line lies west of the true meridian in spring, and east of it in autumn. At the solstices the correction disappears.

For the longitudes of Middle Europe, and for observations made between 8 and 10 A.M. and 2 and 4 P.M., the correction may be written $= 0{\cdot}27\epsilon$, with an accuracy always within 1 m. of arc.

Let the declination of the sun have increased $\Delta\delta$ between its two passages through the altitude h, and by this means the second azimuth have been found ΔA too great. The relation $\Delta\delta \,.\, \cos\delta = \Delta A \,.\, \cos\phi \cos h \sin A$ is found between $\Delta\delta$ and ΔA by differentiation of the equation 1 (**87**). Substituting $\cos\delta \sin\tau$ for $\cos h \sin A$ according to equation 3 (**87**), we have $\Delta\delta = \Delta A \,.\, \cos\phi \sin\tau$. The arithmetical mean of the two observations must be corrected by $\frac{1}{2}\Delta A = \frac{1}{2}\Delta\delta/(\cos\phi \sin\tau)$. We have only to write further $\frac{1}{2}\Delta\delta = \frac{1}{24}\epsilon t$, and $\sin\tau = \sin 15t$ to obtain the above expression.

IV. *From the Observation of the Sun at Noon.*—If the true time (**92**) is known, the meridian is obtained by observation of the sun's centre at 12 h. true solar time (= mean time of the place minus the equation of time, Table 31). The theodolite is directed to the east or west edge of the sun, and the observed azimuth is corrected eastwards or westwards by

$$\Delta = \rho/\sin(\phi - \delta)$$

Here ρ denotes the semidiameter (Table 33), and δ the

declination of the sun, and ϕ the altitude of the pole (latitude of place).

For the meridian, the altitude circle of the sun's edge, and the semidiameter of the sun to its point of contact with the altitude circle, form a right-angled triangle with the hypothenuse $\phi - \delta$, of which the small side ρ is opposite the angle Δ. Therefore $\sin \Delta : 1 = \sin \rho : \sin (\phi - \delta)$. For $\sin \Delta$ and $\sin \rho$, we may write Δ and ρ.

90.—Determination of the Altitude of the Pole for any Place.

I. The geographical latitude or the altitude of the pole at any place is most easily deduced from the observed altitude of a heavenly body at its culmination. If the meridian is already known (**89**) the passage of the body over the meridian is simply observed, otherwise the object is followed with the theodolite in the neighbourhood of the meridian, and the highest (or lowest) position of the telescope is read off.

The observed altitude must be diminished by the amount given in Table 34 for refraction by the atmosphere. If the altitude so corrected is h, and the declination of the body is δ (Table 35), the altitude of the pole is—

$$\phi = 90 - h + \delta \text{ or } \phi = 90 + h - \delta$$

according as it was the upper or lower culmination that was observed.

The measurements are most conveniently and accurately made on the pole-star on account of its slow motion.

To know beforehand the time of culmination of a star, the sidereal time at noon is deducted from the right ascension of the star (Table 35). This gives the time of the upper culmination reckoned in sidereal time from noon. The sidereal hour = 0·9973 mean hours.

The sidereal time is taken from Table 31. On account of the periodical variation of the vernal equinox which is corrected by leap year; and further, since noon is later the farther west a place is, the table cannot be the same for all

years and for all places. If the sidereal time corresponding to the mean time T is required for a place l° E. long. from Greenwich, we must therefore use as argument in the table, not T, but the corrected value (expressed in fractions of a day)—

$$T + k + \frac{1}{360}(15^\circ - l)$$

k has a different value every year, which is found in Table 32.
l may be taken from Table 30, or from a map.

If T is the "unit time" of Middle Europe, only $T+k$ must be taken.

II. Without moving the horizontal circle of the theodolite the *two heights of the pole-star* are observed at which it passes the vertical wire in the course of one circuit round the pole. The mean is taken, which after correction for refraction gives the altitude of the pole.

III. *A single observation* of the pole-star at an approximately known time gives the altitude of the pole since $(90 - \delta) \cos \tau$ (compare **89**, 2) may usually be taken with sufficient exactness as the vertical height of the star above the pole.

On the declination of the sun compare p. 401 and Table 31. Necessarily the position observed in this case by observation of the upper or lower limb of the sun must be corrected by the sun's semidiameter (Table 33).

91.—Determination of the Rate of a Watch or Clock, or keeping True Time.

Two determinations of time (**92**) give, of course, the rate of the clock used in making the observations. More simple, however, and often more accurate, are the observations of a heavenly body at some definite azimuth.

I. *Observations on Fixed Stars.*—For this purpose any telescope which is provided with cross-wires and is movable on a horizontal axis can be used. The azimuth to be employed is defined when at any particular place a distant mark is used

to adjust the telescope. Observations near the meridian are best.

It is still simpler and easily accurate to 1 sec. to use the appearance or disappearance of a fixed star behind a distant object as observed with the naked eye. As the fixed point for the place of the eye, a window bar or any similar object is sufficiently accurate when the other object is distant 100 yards or so. Hot chimneys are unsuitable objects behind which to observe the disappearance of stars.

Of course, it is best to choose stars near the equator. Between two passages of a fixed star through the same point a sidereal day has elapsed which is 235·9 seconds = 3·932 min. = 0·06553 hrs. = 0·002730 day shorter than the mean day.

II. *Observations on the Sun.*—Two successive passages of the sun over the meridian give the length of the mean day, regard being had to the daily variation of the equation of time (Table 31, and Bremiker's *Five-Figure Logarithms,* p. 137). It is not here necessary that the meridian should be quite accurately determined. An error of 1° makes the day as observed at most about 2 secs. in error. At the equinoxes and solstices this uncertainty is the smallest.

A telescope on a horizontal axis is used for the observations, the time of the passage of the two edges of the sun over the wire being noted. Where moderate accuracy only is required, even the shadow of a plumb-line, or the image of the sun thrown by a narrow opening, is sufficient. The time is noted when this shadow or the sun's image is bisected by a mark on the floor, or on a wall opposite. A good sun-dial also allows of pretty accurate timing of a clock over longer intervals.

True time once obtained may be kept by these simple means.

92.—Determination of the Time from Altitudes of the Sun.

I. *From a Single Altitude.*—For a place of known geographical longitude and latitude the simplest method of

determining the time is afforded by observations of the altitude of the sun above the horizon, which is measured by the sextant or theodolite. Those times are the most suitable for the observations in which the ascending or descending motion of the sun is as rapid as possible,—when, therefore, its position is directly east or west. The nearer noon, the less accurate is the determination.

Let

ϕ = the geographical latitude of the place, or the altitude of the pole;
δ = the declination of the sun at the time of observation (see following page);
h = the true altitude of the sun's centre;

then the sun's hour angle t, or the "true solar time" of the observation, will be got by the formula—

$$\cos t = \frac{\sin h - \sin \phi \, . \, \sin \delta}{\cos \phi \, . \, \cos \delta}$$

The angle t is in the first instance obtained from the trigonometrical tables in ordinary angular measurement. If expressed in degrees, it must be divided by 15 to give solar time in hours. It is in the morning negative, in the afternoon positive.

In the spherical triangle, which is formed by the meridian and the altitude and declination circles of the body, and has the sides $90 - \phi$, $90 - h$, and $90 - \delta$, while the hour angle t is opposite the side $90 - h$, we must have

$$\sin h = \sin \phi \sin \delta + \cos \phi \cos \delta \cos t.$$

Correction of Observed to Actual Altitudes.

The observed place appears too high on account of the atmospheric refraction, and must be corrected by subtracting from it the refraction given in Table 34.

Further, the observations are not made directly on the centre, but on either the upper or lower edge of the sun. The position of the centre is found by adding or subtracting, as the case may be, the radius (Table 33).

If, however, the horizon point of the altitude circle is not already known, but has to be eliminated by reversal, a second observation being made with the instrument turned through 180°, the sun's diameter may be eliminated also by observing first one edge and then the other. It is necessary to make the two observations quickly following each other if we want to use the mean of the two observations as the altitude of the sun's centre at the mean of the times of observation, since the rate of the sun's rising is not uniform.

Some Geographical Latitudes are given in Table 30. The latitude of a place may be taken from a good map to about 0°·01. For the method of determining it see (**90**).

The Sun's Declination for the time of the observation is interpolated from Table 31 after correction by $+k$ (Table 32) or $+k+(15-l)/360$ as required. Compare **90**, end of I. An error of 3 min. in the time gives at most an error of 0°·001 in δ.

Mean Time.—The equation of time (Table 31) must be added to the true solar time t to give the mean time of the place; and for the Middle Europe "unit time," in addition to the equation of time $+(15-l)\times 4$ min. (For Greenwich time this becomes $-4l$ min. for E. and $+4l$ min. for W. longitudes.)

Other Bodies.—Instead of the sun, another heavenly body of known declination and right ascension (Table 35), and which is not too near to either horizon or pole, may be made use of. The value of t calculated from the formula given (p. 400) is then the hour angle of the body. If to this the right ascension be added, the sidereal time of the observation is obtained, from which the mean time may be found by Table 31, or more accurately from the astronomical almanacs.

The tables and directions here given neglect corrections of less than 0°·01.

II. *From Observations of Equal Altitudes.*—If the two points of time are observed at which a heavenly body before and after its culmination passes the horizontal wire of a telescope fixed at a constant altitude, the arithmetical mean of the times is the moment at which the body culminated. The true time of the culmination is found from the Tables.

For a Fixed Star this coincides with its southern meridian passage. At this instant the right ascension of the star (Table 35) is therefore equal to the sidereal time, and from it the civil or mean time may be obtained from Table 31, or the nautical almanacs.

If *the Sun* is observed, the greatest altitude coincides with the meridian passage (*i.e.* the true noon) at the solstices. In general, however, a correction is necessary on account of the daily alteration of the sun's declination, in consequence of which the sun attains its greatest altitude somewhat after noon in the first half of the year, and somewhat before noon in the second half. If as before

ϕ = the polar altitude (or geographical latitude);
δ = the declination of the sun (Table 31);
ϵ = the daily alteration of the declination in degrees (Table 31, or Bremiker's *Five-Figure Logarithms*, p. 139); and lastly
τ = half the interval of time in hours between the two observations (therefore $\pm 15\tau$ the hour angle of the sun in degrees);

this correction in seconds of time amounts to

$$10\epsilon\tau(\tan\phi - \tan\delta\cos 15\tau)/\sin 15\tau.$$

The mean time of the place is found by adding to the "true solar noon" the value given by the equation of time (Table 31).

Let t be the hour angle of the sun at the observation. But for the change of declination the absolute value of t before and after noon would be the same. If, however, between the first and the second passage through the altitude h the declination is increased by $\Delta\delta$, the value of t at the second passage will be found too great by Δt, for which, by differentiation of equation 2 (**87**), the relation is obtained

$$0 = \Delta\delta\,.\,(\sin\phi\cos\delta - \cos\phi\sin\delta\cos t) - \Delta t\,.\,\cos\phi\cos\delta\sin t.$$

Therefore $\Delta t = \Delta\delta(\tan\phi - \tan\delta\cos t)/\sin t.$

It is obvious that to reduce the arithmetical mean of the two observed times of passage to that of the passage through the meridian, the correction $\frac{1}{2}\Delta t$ must be made. If we consider further that $t = 15\tau$, and that $\frac{1}{2}\Delta t$ in degrees of arc $= \epsilon\tau/24$, or in seconds of time $= 86400/360\,.\,\epsilon\tau/24 = 10\epsilon\tau$, we obtain the above expression.

As to instrumental means this method of time determination is very simple, requiring, in addition to a clock of uniform rate, only a telescope with a vertical axis of rotation (**88**, I.) without any graduation. For ordinary purposes no account need be taken of the atmospheric refraction, and in observations of the sun only the upper or lower edge is observed each time without our being obliged to reduce the altitude to that of the centre.

To make the determination as accurate as possible, the heavenly body should be observed as far from the meridian as possible.

On the simple methods of keeping a knowledge of the true time when it has once been found, compare **91**.

THE ABSOLUTE SYSTEM OF ELECTRICAL MEASUREMENT.

Every kind of magnitude requires for its measurement—that is, its numerical expression—some unit of the same nature as itself. This unit is at first arbitrary, and may be defined, for many kinds of magnitude, as a preserved original measure (scale, standard); but in other cases, as, for instance, velocity, or quantity of heat, or electricity, such a definition is impossible. Hence, such magnitudes are expressed by means of geometrical and physical laws, in terms of other quantities which can be so defined; as, for instance, *velocity* by length and time, a quantity of *heat* by a mass of water and temperature, and a quantity of *electricity* by the force it exerts upon another quantity. As distinguished from the arbitrary or primary measures, we may call the latter "derived measures."

The introduction of such measures, unavoidable in the first instance, will be seen on further consideration to be also very advantageous. For it is obvious that the diminution of the number of arbitrary primary measures is in itself an advance, while the new units may be so chosen that they give the simplest form to the mathematical or physical law which is used to define them. For instance, the space l traversed by a moving body is universally proportional to the velocity u and the time t, or $l = \text{Const. } ut$—the numerical value of the constant depending on the unit selected. Should we take as unit of velocity that of a falling body at the end of the first second, this constant $= g$. If, however, we take as unit the velocity with which the body traverses unit distance

in unit time, the constant becomes 1, and the law will take its simplest form, $l = ut$.

The geometrical relations are similarly simplified if we employ for the measurement of area and contents, instead of arbitrary units, the square and cube of the unit of length, an advantage of which science has always availed itself, but which is not yet fully carried out in common life. In this manner each "derived unit" serves to eliminate the constant of a natural law.

Among the objects to which preserved elementary standards are inapplicable we may count almost all magnetic and electrical quantities, and hence we have here a specially prominent application of the system of derived units. This application was carried out by Gauss and Weber, who showed that all these quantities might be expressed in units of length, mass, and time. Units deduced in this manner are specially called "absolute" measure.*

The choice of primary units of length, mass, and time is in the first instance entirely arbitrary. If, however, water is taken for the determination of the unit of density, the unit of volume of water is fixed as the unit of mass, and then necessarily we have for units—

of length: millimeter, centimeter, decimeter, meter;
of mass: milligram, gram, kilogram, 1000 kilo.

It must also be distinctly understood that, in the absolute "dynamic" system of measures, a gram means the *mass* of 1

* The term "absolute" was first applied in this manner to the unit of intensity of terrestrial magnetism defined by Gauss. In opposition to the arbitrary practice, previously common, of taking the intensity at London as unity, and making other observations merely relative to this, Gauss gave in his *Intensitas vis magneticæ terrestris ad mensuram absolutam revocata* an absolute (that is, not a merely comparative) unit for terrestrial magnetic intensity, deduced from the primary units of length, mass, and time only, and applicable to magnetic quantities in general. In a similar manner W. Weber, in supplying the need of independent, and not merely comparative measures, for the various electrical quantities, has retained the same designation.

The name "absolute measure" has now become a scientific phrase of determinate meaning, and must therefore be unconditionally retained, although it must be admitted that the term "derived measure" hits more exactly the essential point of the system (*Brit. Assoc. Rep.* 1863, p. 112).

cubic centimeter of water; whilst, in popular language, grams, etc., are usually spoken of as *weights*. For example, the moment of inertia of a small body of $m^{\text{mgr.}}$ or grm., and distant $a^{\text{mm.}}$ or cm. from an axis of rotation, is, in the absolute system, $= a^2m$, and not a^2m/g. On the other hand, the weight of this body is mg, and on this account the moment of rotation, which it experiences from the attraction of the earth, at a horizontal distance a from an axis of rotation $= amg$, where by g we denote the acceleration by gravity, measured in mm. and sec., which at latitude $45° = 9806$. To avoid confusion it is advisable, when the gram is used as the name of a weight, to speak of it as a gram weight.

It is worth mentioning that Gauss in his first treatise on this subject (*Erdmagnetismus und Magnetometer*, Schumacher's Jahrbuch, 1836; Gauss, *Werke*, vol. v. p. 329) defined the absolute magnetism of a bar by means of the unit of *weight*, and that it was only at a later date that he assumed the gram as a *mass*.

If a general answer be desired to the question whether grams, etc., have to serve as units of mass or weight, there can be little doubt as to the scientific reply:—That, as the weight of a body is clearly entirely indeterminate, and is even variable on the earth's surface to the extent of $\frac{1}{2}$ per cent, the weight of a body can never serve as a unit of weight. It would also be wrong to say that as unit of weight we take a cubic centimeter of water at 45° latitude; since then a set of weights would have to be specially adjusted for each degree of latitude. What is really meant by the phrase "set of weights" is nothing but a set of masses; and a weighing with an ordinary balance is no measurement of weight, but one of mass. The weight, that is, the force with which a body is attracted by the earth, is obtained by the measurement of velocity of falling; as, for instance, by the time of oscillation of a body suspended by a thread.

In fact, also the aim of weighing is generally measurement of mass. The chemist, the merchant, and the doctor, have nothing to do with the pressure of a body on what supports it, but solely with its mass, for to this its chemical power, its nutritive or its money value, is proportional.

In physics the conception of the gram as a unit of mass is now almost universal. The scattered cases (such as elasticity, capillarity, etc.) in which it is still used in another sense from old habit, and, it must be admitted, with practical convenience, will gradually die out. As regards the expressions "specific gravity" and "density," although the one belongs to the dynamic and the other to the static system of measurement, their numerical values are identical, since both adopt water as unit.

Dimensions.—All magnitudes may be expressed as functions of length $[l]$, mass $[m]$, and time $[t]$; a velocity, for instance, as length / time, a volume as length^3, a force as length $\times$ mass / time^2. These "dimensions" (after Maxwell and Jenkin, *Brit. Assoc. Rep.* p. 132, 1863) will be given for each magnitude in the following article in square brackets. Thus the dimensions of a velocity are $[lt^{-1}]$, of a volume $[l^3]$, of a force $[lmt^{-2}]$, etc.

The "dimension" is a very useful test for the correctness of a physical equation; since, if for each of the magnitudes which it contains, we substitute their dimensions, the same must be found for both sides.

The work, for instance, done in time t, by an electrical current i, of which the *E.M.F.* or potential difference in a given conductor $= E$ is $L = Eit$. The dimensions (Table 28) of E are $[l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-2}]$, of i $[l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}]$; therefore those of Eit are $[l^2mt^{-2}]$, which are the dimensions of Work.

The "dimensions" give the power to change from one group of primary units to another, say from mm., mg., to cm., g.* For if a

* For many years the system founded on the fundamental units mm., mg., and sec., was universally used. It was in England, where the British Association, and especially J. C. Maxwell and Sir W. Thomson, have done so much for the knowledge and dissemination of the absolute measurements, that cm. and g., instead of mm. and mg., were introduced by general agreement.

In fact mm. and mg. are inconveniently small units for many purposes, and the magnetic and electrical measures derived from them have in some cases this disadvantage to a tiresome extent. There is some advantage in this respect in using cm. and g. It is nevertheless doubtful whether it was expedient to destroy the uniformity before existing in the absolute system, since the disadvantages are but partially removed. The resistances and electromotive forces, for instance, which occur in practice are even in the cm. g. system expressed by quantities of many millions of the units. On the other hand, the galvanic unit of current reaches such a magnitude that most currents which are used are expressed as only small fractions of it. And if the physics of the future has more to do with the absolute numbers of atoms than we have at present, even the cubic mm.

primary unit occurs in the deduced unit in the pth power, the deduced unit is changed in the ratio n^p if the primary be changed in that of n. The numerical value of the magnitude thus expressed will be changed in the ratio n^{-p}. Thus the number representing a velocity l/t will, by the change from mm. to cm. as unit of length, be changed itself in the ratio 10^{-1}; by change from sec. to min. in that of 60^{+1}. The numerical value of a force lm/t^2, expressed in cm. g., instead of mm. mg., will be diminished in the ratio $10^{-1} \,.\, 1000^{-1} = \frac{1}{10000}$ (Table 28).

The "technical" system of electrical measurement introduced by the British Association, in which the ohm, ampere, volt, and farad are the units of resistance, current-strength, electromotive force, and capacity respectively, is related to a system of primary units, which beside the second as unit of time, contains the earth-quadrant $= 10^9$ cm. as unit of length, and the 10^{11} part of a grm. as unit of mass. If the dimensions of a magnitude $= l^\lambda \,.\, m^\mu \,.\, t^\tau$, the unit of the "technical" system is $10^{9\lambda} \,.\, 10^{-11\mu}$ greater than that founded on cm. g.

A current-strength, for instance $= l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}$, therefore the ampere $= 10^{\frac{9}{2}} \,.\, 10^{-\frac{11}{2}} = 10^{-1}$ cm. g. current units (compare No. 19). The unit of work 1 volt-ampere-sec. or watt-sec. $= [l^2mt^{-2}]$, or $10^{18} \,.\, 10^{-11} = 10^7$ absolute cm. g. units of work.

The prefixes mega- or micro- (*e.g.* megohm or microfarad) denote 10^6 times larger or smaller units.

Measures of Space and Time.

(1.) *Surface* $f = [l^2]$.—As the unit of surface the square of the unit-length is used.

(2.) *Volume* $v = [l^3]$.—Unit-volume is the cube of the unit of length.

(3.) *Angle* ϕ.—An angle is, in mechanics, equal to the arc

would be inconveniently large. Every *consistent* system of measurement must give rise to awkward numbers.

However, the gram is more convenient to have to do with than the mg., and the most easily understood, and frequently occurring magnitudes of mechanics, force, work, moment of inertia, and so on, are usually expressed in much more convenient numbers in the cm. g. system. The extent to which it is used and the authority of the British Association also lend it considerable weight. Table 28 gives the ratios of the units in the two systems.

which subtends it divided by the radius. That angle, therefore, is the unit of which the arc is equal to the radius ($=57°{\cdot}296$). Small angles are numerically equal to their sines or tangents. Dimensions $= l/l = 1$ (*i.e.* is independent of the fundamental units).

(4.) *Velocity* $u = [lt^{-1}]$.—Velocity is measured by the space passed over divided by the time occupied. The unit velocity, therefore, is that of a point traversing unit-length in unit time. *Angular velocity* of rotation is measured by the angle traversed by a radius in unit time. Dimensions $= [t^{-1}]$.

(5.) *Acceleration* $b = [lt^{-2}]$.—If the velocity increases by the quantity u in the time t, the body experiences an acceleration $b = u/t$. The unit is therefore that acceleration which produces the unit velocity in unit time.

The acceleration by gravity amounts to 9806 mm. sec.$^{-2}$ or 9·806 m. sec.$^{-2}$ or $9{\cdot}806 \times 60^2 = 35302$ m./min.^{2}.

Mechanical Measures.

(6.) *Force* $k = [lmt^{-2}]$.—By an elementary law of mechanics, the force k which communicates the velocity u to a body of mass m in time t, is directly proportioned to the quantities m and u, but inversely to t; or $k = C \,.\, um/t$, the constant C being dependent on the unit selected. Taking $C = 1$, with unit u, t, and m, k must also $= 1$, and therefore *unit of force* is that force which in unit time communicates unit velocity to a unit mass.

The force exerted on 1 mgr. by the earth's attraction $= 9806$ mm. mg. sec.$^{-2} = 0{\cdot}9806$ cm. g. sec.$^{-2}$. The absolute cm. g. unit of force or "dyne" (Clausius) is therefore a little greater than the attraction of the earth for 1 mg.

(6*a*.) *Pressure* $d = [l^{-1}mt^{-2}]$.—If forces are uniformly distributed over a surface, the total force acting vertically on the unit of surface is called pressure.

The pressure of 1 cm. of mercury is $= 13{\cdot}596 \times 980{\cdot}6 = 13332$ cm.$^{-1}$ g./sec.^{2} or dyne/cm.^{2}; of 1 atm. $= 76 \times 13332 = 1013200$ cm.$^{-1}$ g./sec.^{2}

(7.) *Work, Energy, Vis viva, Heat,* $L = [l^2mt^{-2}]$.—Work is performed when the point of application of a force is moved by it.

We express work as the product of force and distance, $L = kl$, and therefore *unit work* is performed when a point, acted on by unit force, is moved by it through unit distance.

Potential Energy of a body or a system is the sum of the work which the body or system can perform by movement under the influence of the acting forces.

Kinetic Energy or *vis viva* of a mass m moving with velocity u is $\frac{1}{2}mu^2$, and is of the same value as *work*.

The *unit quantity of heat* is that which is equivalent to the unit of work.

In raising 1 kg. 1 meter (technical "kilogrammeter"), the work $1000 \times 980{\cdot}6 \times 100 = 98060000$ cm.^{2} g./sec.^{2}, is performed (1 "erg" of Clausius).

The unit of heat ordinarily used, or "water-gram-calorie," which heats 1 grm. of water from 0° to 1°, and which is equivalent to 428 gram-weight × meters of work, is in absolute measure $428 \times 980{\cdot}6 \times 100 = 42000000$ cm.^{2} g./sec.^{2} absolute work-equivalents of heat. An increase of volume of v cm.^{3} under constant pressure of p cm. of mercury yields the work $vp \times 13332$ (compare 6*a*); if $p = 1$ atm., $v \times 1013200$ ergs.

Work done by Expansion of Gases.—Let a volume of gas V be warmed at constant pressure d from the absolute temperature T to $T+1$. The dilatation is V/T, and the external work $Vd/T = R$. If the mass of gas is 1 g., R is the "gas-constant"; for air, it equals $773{\cdot}4 \times 1013200/273 = 2870000$ cm.^{2} g./sec.^{2}. $773{\cdot}4$ cm.^{3} is the vol. of 1 grm. air at 0° C. and 760 mm.

Work of Gasification.—1 gram-molecule (for instance, 2 g. hydrogen) has, at 0° and 1 atm., the volume $V = 773{\cdot}4 \times 28{\cdot}9 = 22350$ cm.^{3}. The external work of gasification is $22350 \times 1013200 = 22650 \times 10^6$. At the absolute temperature T, this becomes $22650 \times 10^6 \times T/273 = 830 \times 10^5 T$ cm.^{2} g./sec.^{2}. The amount of external heat consumed in gasification under constant pressure is therefore $830 \times 10^5 T/(420 \times 10^5) = 2{\cdot}0T$ water-g.-calories.

(8.) *Moment of Rotation* $P = [l^2mt^{-2}]$.—Taking the moment of rotation P as the product of a force k into the length of its leverage l (that is, its distance from axis of rotation), $P = kl$; *the unit of moment of rotation* is given by unit force acting through a lever of unit-length.

(9.) *Directive Force* $D = [l^2mt^{-2}$.—If a body, movable round a fixed axis, has a stable position of equilibrium, a moment of rotation P is exerted on it in any other position, which, for a

small angle of deflection ϕ, is proportional to ϕ. The constant ratio $P/\phi = D$ is the directive force exerted on the body (the angle being measured in arcual units).

The directive force of a pendulum moved by gravity, of which the mass $m = 1$ k. and the length $l = 1$ m. from the point of suspension to the centre of gravity, is therefore $1000 \times 100 \times 980{\cdot}6 = 98060000$ cm.^{2} g. sec.$^{-2}$, for the moment of rotation for a deflection $\phi = lgm \sin \phi$, and for small angles, ϕ may be taken as $= \sin \phi$.

The directive force due to weight of a bifilar suspension (**54**) with a separation of threads of 10 cm. and a length of thread of 200 cm. is for the mass of 1000 g. $= \frac{1}{4} 10 \times 10/200 \times 1000 \times 980{\cdot}6 = 122600$ cm.^{2} g. sec.$^{-2}$.

(10.) *Moment of Inertia* $K = [l^2 m]$.—Taking the moment of inertia K of a mass m at the distance l from its axis of rotation, $K = l^2 m$, or, if several masses are present, $K = \Sigma l^2 m$; therefore *the unit of moment of inertia* is represented by a point, of unit mass, at unit distance from an axis of rotation.

The moment of inertia of the above pendulum is therefore $100^2 \times 1000 = 10^7$ cm.^{2} g. A rectangular bar of 10 cm. length and 1 cm. breadth, and weighing 50 g., has in relation to its centre the moment of inertia (**54**) $\frac{1}{12}(10^2 + 1^2) \times 50 = 421$ cm.^{2} g.

Moment of inertia K, directive force D, and time of oscillation t for small arcs, are connected by the equation $\dfrac{t^2}{\pi^2} = \dfrac{K}{D}$, as the dimensions themselves show, since $l^2 m$ divided by $l^2 m t^{-2}$ gives the square of a time.

(10*a*.) *Modulus of Elasticity* $\eta = [l^{-1} m t^{-2}]$.—If we write $\lambda = 1/\eta \,.\, k L/l^2$ for the elongation λ which is produced in a bar of length L and of sectional area l^2 by a tensile strain K, the modulus of elasticity η is the force which, acting on a section of unit area, would double the length of the bar.

The kg.-weight/mm.^{2} moduli of elasticity in practical use must be multiplied by 98100000 to bring them to those of the absolute cm. g. system. Compare p. 129.

ELECTROSTATIC MEASURE.

(11.) *Electrical Quantity* $\epsilon = [l^{\frac{3}{2}} m^{\frac{1}{2}} t^{-1}]$.—Two quantities of electricity, ϵ, ϵ', considered as concentrated in points, and at

the distance l, repel each other with the force $k = C \cdot \epsilon\epsilon'/l^2$ in which the numerical value of C depends on the unit selected. Putting the constant $C = 1$, and so giving the law its simplest form, $k = \epsilon\epsilon'/l^2$, the so-called mechanical *unit of electrical quantity* is that quantity which repels an equal quantity at unit distance with unit force.

For the square of a quantity of electricity is given as a force (lmt^{-2}, compare 6) multiplied by the square of a length, therefore the dimensions of a quantity of electricity in mechanical units $= \sqrt{l^3mt^{-2}} = l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-1}$.

Lines of Force (Faraday).—The action of electrical quantities may be represented by lines. Each unit of electricity gives out 4π lines of force. The direction of these lines at any place in an "electric field" gives the direction of the latter; their density—that is, their number in a bundle of unit section, gives the strength of the field or force exerted on a unit quantity of electricity at that place.

(12.) *Electrostatic Potential or "Pressure"* $V = [l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}]$.—When we have to do with masses which attract or repel as the inverse squares of their distances, the potential function or potential of these masses on any point in their neighbourhood is that expression the variation of which in any direction gives the force exerted at that point in that direction on unit mass. By variation we mean the amount by which the expression diminishes when we pass from the point considered to another near to it, divided by the distance between the points; in short, the negative differential coefficient of the expression in the given direction. Therefore the potential of the quantity of electricity ϵ on a point distant l is given by ϵ/l; if there are several quantities ϵ_1, ϵ_2, ... present, their potential on a point distant l_1, l_2 ... from them is $\epsilon_1/l_1 + \epsilon_2/l_2 + \ldots$. The *unit of electrostatic potential* is therefore the potential of the unit quantity of electricity on a point at unit distance.

The potential has further the important significance that it measures the quantity of work which is done by the electric forces when the unit quantity of electricity is removed from that place to a very great distance from the quantity producing the potential.

(13.) *Electrostatic Capacity* (electrostatically measured) $c=[l]$.—In order that a quantity of electricity ϵ may be in equilibrio on a conductor, it must be so distributed that its potential V is equal on all points of the conductor. Let the surroundings of the conductor be at zero potential. Then the potential is proportional to the quantity of electricity on the same conductor; $\epsilon=c\,.\,V$. The ratio $c=\epsilon/V$ is called the electrostatic capacity of the conductor.

The capacity of a sphere is equal to its radius, for the quantity of electricity ϵ uniformly distributed over a spherical surface of radius r exerts on the centre, and consequently on every point of the sphere, the potential ϵ/r.

The potential of a charged conductor is therefore measured by the quantity of electricity which it communicates to a very distant sphere of unit radius connected with it by a fine wire.

That conductor has the *unit capacity* which is charged to unit potential by unit quantity of electricity, as, for instance, a sphere of radius 1.

(13*a*.) *Dielectric Constant, or Specific Inductive Capacity* $K=[l^0m^0t^0]$. A condenser of surface f, at the relatively small distance l from a parallel surface connected to earth, has the capacity $K\,.\,f/4\pi l$ where K is dependent on the nature of the medium separating the surfaces (dielectric of Faraday), and is called the dielectric constant, or specific inductive capacity of the medium. It is independent of the primary units.

Magnetic Measure.

(14.) *Free Magnetism, or Strength of Magnetic Pole* $\mu=[l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-1}]$.—Exactly as above for electrical quantities, we may write the elementary law of the interaction of two hypothetical quantities $\mu\ \mu'$ of free magnetism (or two magnetic poles of the nature of points, of strength μ and μ') which at the distance l repel each other with the force k, as $k=\mu\mu'/l^2$, and so obtain as *unit quantity of free magnetism* (or *strength of unit pole*) that quantity or pole which exerts unit force on a similar one at unit distance.

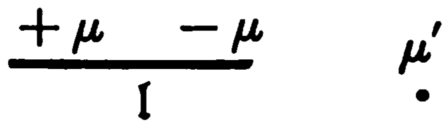
(15.) *Magnetism of Bar, or Magnetic Moment* $M=[l^{\frac{5}{2}}m^{\frac{1}{2}}t^{-1}]$.—Each magnet has equal quantities of free positive and nega-

tive magnetism. The simplest bar-magnet would consist of two opposite poles of the nature of points, and of equal strength. If $\pm\mu$ be the quantity of magnetism which is contained in each pole, and l the distance between them, the action of the bar at a distance will be proportional to $l\mu$, which is the magnetic moment, or, shortly, the magnetism of the bar. A magnet which consists of two poles, with the quantity ± 1 of free magnetism (or of unit strength), and separated by unit distance, represents the unit of strength of a bar-magnet.

The unit [cm. g.] is 10000 times greater than [mm. mg.]

The Specific Magnetism of a Bar is the ratio of the magnetic moment to the mass or volume of the magnet. It amounts in very thin bars to at most about 100 cm.$^{\frac{5}{2}}$ g.$^{-\frac{1}{2}}$ sec.$^{-1}$ for each grm. of steel.

Action at a Distance. First Position.—The magnetic pole μ' lies in a line passing through the two poles, at a distance L from the centre of the magnet μ l.



The total force exerted on μ' is the difference between that exerted by the two poles, or

$$k=\mu\mu'[1/(L-\tfrac{1}{2}l)^2-1/(L+\tfrac{1}{2}l)^2]=\mu\mu'\,.\,2Ll/(L^2-\tfrac{1}{4}l^2)^2$$

$l\mu=M$, the magnetism of the bar, and therefore

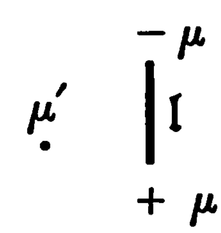
$$k=2M\mu'L/(L^2-\tfrac{1}{4}l^2)^2=2M\mu'/L^3\,.\,(1-\tfrac{1}{4}l^2/L^2)^{-2} \qquad (1)$$

or by expansion in series (p. 10, formula 1)

$$k=2M\mu'/L^3\,.\,(1+\tfrac{1}{2}l^2/L^2+\tfrac{3}{16}l^4/L^4+\;.\;.\;.)$$

It is desirable to work at distances sufficiently great to admit in any case of the third term being neglected. If L be sufficiently large as compared to l for $\frac{1}{2}l^2/L^2$ to be neglected compared to 1, the expression becomes simply $k=2M\mu'/L^3$.

Second Position. — The magnetic pole μ' is placed on a line perpendicular to the axis of the magnet, and passing through its centre, and at the distance L from the middle of the magnet. The dissimilar pole exerts an attractive force $=\mu\mu'/(L^2+\frac{1}{4}l^2)$, and the similar pole a repulsion of like amount. Both forces are resolved, according to the parallelogram of forces, into a single force acting parallel to the axis of the bar.



$$k=\mu\mu'/(L^2+\tfrac{1}{4}l^2)\,.\,l/\sqrt{L^2-\tfrac{1}{4}l^2}=M\mu'/L^3\,.\,(1+\tfrac{1}{4}l^2/L^2)^{-\frac{3}{2}} \qquad (2)$$

for which we may write

$$k = M\mu'/L^3 . (1 - \tfrac{3}{8}l^2/L^2 + \tfrac{15}{128}l^4/L^4 + \ldots)$$

At a very great distance L, the expression becomes $k = M\mu'/L^3$.

If we replace the magnetic pole μ' by a short magnetic needle, at right angles to the direction of the force, and of the length l', and of which each of the poles has the strength μ', a couple will be produced exerting a moment of rotation $2kl'/2 = kl$ upon it. Since $\mu'l'$ is the magnetic moment of the needle M', the moment of rotation exerted on it by another magnet M at the distance L (great compared with the length of the magnets) will be—

In the first position to it, $P = 2MM'/L^3$.
In the second position $P = MM'/L^3$.

Hence we may also define the unit of bar-magnetism as follows:—

> The unit of bar-magnetism is possessed by a bar which exerts on a similar bar at the (great) distance L, in the *first position* (compare previous page), the moment of rotation $2/L^3$, or in the second position, that of $1/L^3$.

If the length l' of the needle is not small enough for l'^2 to be neglected as compared to L^2, the following factors must be included in the expressions for k, *i.e.*

In the *First Position* $1 - \tfrac{3}{4}l'^2/L^2$.
In the *Second Position* $1 + \tfrac{3}{2}l'^2/L^2$.

If the deflected magnet makes an angle ϕ with the direction of the force, the moment of rotation will obviously be obtained by multiplying the above result by $\cos\phi$.

What is here indicated for ideal magnets, with points for poles, is also true of the actual. For, in action at a distance, in slender magnets at distances at which l^4 may be neglected as compared to L^4, there are two mean points in which we may consider the positive and negative magnetism to be concentrated. Ordinarily, the distance apart of these "poles" (the "reduced length") is about $\tfrac{5}{6}$ that of the bar. If this is not assumed, the length of the magnet must be eliminated by the Gauss method (**59**, II.), by observations at two distances. Calculation by the unabridged formulæ 1 and 2 (see previous page) is more accurate than that by expansion in series with a correcting term, and the expressions for M/H on p. 244 are so obtained.

Separation of a Magnet in Components.—A magnet M which

forms the angle α with the connecting line may be considered as regards action at a distance as consisting of two bars of the strengths $M\ cos\ \alpha$ and $M\ sin\ \alpha$ acting from the first and second positions respectively.

(16.) *Magnetic Intensity of a Place, or Strength of a Magnetic Field* $H = [l^{-\frac{1}{2}} m^{\frac{1}{2}} t^{-1}]$.—At any place a magnetic pole is usually acted on by a force k proportional to the strength μ of the pole, or $k = \mu H$. The quantity H, which is the force exerted on a unit pole, is the intensity of magnetic force at the place, or, shortly, the magnetic intensity, or strength of magnetic field.

The dimensions are $[k/\mu] = [lmt^{-2}]/[l^{\frac{3}{2}} m^{\frac{1}{2}} t^{-1}] = [l^{-\frac{1}{2}} m^{\frac{1}{2}} t^{-1}]$. Quantities given in mg. mm. must therefore be divided by 10 to convert them to cm. g.

The intensity produced at B by a magnet M at A is given by the right-angled triangle ABC. Let $AD = \frac{1}{3}AC$. Then BD is the direction, and $M \,.\, AB^{-3} \,.\, BD/AD$ is the intensity of magnetic force at B. Proof is easy by separating in components (No. 15, end).

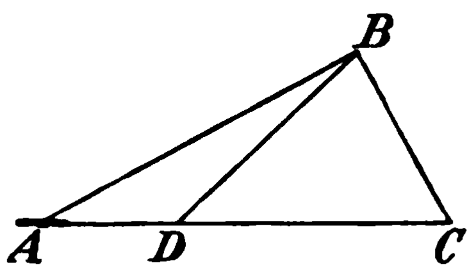


Fig. 91.

The moment of rotation of a magnet perpendicular to the direction of force, and with two poles $\pm\mu$ at a distance l apart, is $2\mu H \,.\, \frac{1}{2}l = \mu l H = MH$, where M is the magnetic moment of the needle. We may therefore define the *unit of magnetic intensity* as that which exerts a unit moment of rotation on a bar of unit magnetic moment at right angles to the direction of the force.

Supposing that the magnet makes the angle ϕ with the direction of the force, we have a moment of rotation $= MH\ sin\ \phi$. But MH is that magnitude which we have previously called directive force, and it determines, therefore, the equation $t^2/\pi^2 = K/MH$ for time of oscillation t, and moment of inertia K. (See No. 10.)

For horizontally oscillating magnets, H is the horizontal component only of the intensity.

Let the horizontal intensity H amount to $0{\cdot}2\ cm.^{-\frac{1}{2}} g.^{\frac{1}{2}} sec.^{-1}$. A thin magnetic bar is 10 cm. long and weighs 20 g. Its moment of

inertia $K = 20 \times 10^2/12 = 167$ *cm.*^{2}*g.* Let the magnetism of the bar be $M = 400$ *cm.*$^{\frac{3}{2}}$*g.*$^{\frac{1}{2}}$*sec.*$^{-1}$; then the time of oscillation is $t = 3{\cdot}14 \sqrt{167/(400 \times 0{\cdot}2)} = 4{\cdot}5$ sec.

The angle through which a short magnetic needle is deflected from the magnetic meridian by another magnet is obtained as follows:—

The magnet M is placed in the "first position" (**59**, II.) to the needle, of which the moment is M' and distance L. If ϕ be the angle of deflection, the moment of rotation exerted by the magnet for this angle $= 2MM'/L^3 . (1 + \frac{1}{2}l^2/L^2)$ *cos* ϕ, which is equal to that, $M'H$ *sin* ϕ exerted by the terrestrial magnetic force. Therefore— *tan* . $\phi = 2/L_3 . M/H . (1 + \frac{1}{2}l^2/L^2)$.

In the "second position" the factor 2 disappears, and instead of $\frac{1}{2}l^2$ we have $-\frac{3}{8}l^2$.

The quantity expressed by η on p. 242 has also the significance that the polar separation of the magnet is represented by $\sqrt{2\eta}$ in the first, and $\sqrt{-\frac{8}{3}\eta}$ in the second position.

Lines of Force.—The magnetic intensity may be represented by lines (Faraday) of which the direction gives that of the force and the density (number per unit of surface measured at right angles to their direction) represents its intensity.

From a magnetic pole $+\mu$ or $-\mu$, $4\pi\mu$ positive or negative lines of force pass into surrounding space. Within an ideal magnet the lines run parallel to the axis from the south to the north pole. The number of lines divided by 4π is the magnetic moment of the magnet per unit of length. From an actual magnet lines of force also pass out laterally, each line passing out representing the quantity of free magnetism $1/4\pi$.

Constant of magnetisation κ ("susceptibility") is the ratio of the specific magnetism (per unit of volume) of a body to the total force which causes the magnetisation. $1 + 4\pi\kappa$ is called "permeability."

Galvanic Measure.

(17.) *Current-Strength. Mechanical or Electrostatic Measure* $i = [l^{\frac{3}{2}} m^{\frac{1}{2}} t^{-2}]$.—This is the electrostatically measured quantity of electricity (No. 11) which passes through a section of the

circuit in unit of time, and hence the *unit of current-strength* is that current in which unit quantity of electricity passes.

(18.) *Chemical Current-Measure.*—Here the *unit* is that current which in unit time effects unit chemical action.

If we knew the absolute number of the atoms in a body this unit of electrical quantity would be most simply determined as that which separated 1 (univalent) atom. So long as we do not know the number of the atoms, and can refer only to the weights separated, this measure is not absolute in the full sense, for the quantity of an electrolyte decomposed by the current is dependent on the nature of the substance; and hence, not only on units of mass, length, and time, but on an arbitrary quantity of the substance employed. Since the decomposition of equivalent weights is proportional (Faraday), and since the chemist takes that of hydrogen as unit, we employ in current-measurement the separation of a unit of hydrogen as unit of chemical action.

(19.) *Electromagnetic (or Weber's) Current-Measure* $i = [l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}]$.—The (transverse) force exerted between a rectilinear portion of a current of length l and of strength i and a magnet pole μ, at a distance L from the current-element, measured at right angles to its direction, is $k = Cli\mu/L^2$. Taking $C = 1$, we may define the *electromagnetic unit of current-strength* as a circular current of which unit length exerts unit force on a unit magnetic pole in its centre.

A closed circular current i of the radius R exerts on a magnetic pole in its centre the force $k = \mu i 2\pi R/R^2 = \mu i \,.\, 2\pi/R$. Compare **64**.

Electrodynamic Current-Measure is identical with electromagnetic, if Ampere's law be stated as follows:—Two similarly directed currents i and i' in the straight conductors l and l', at the (relatively great) distance L apart, attract each other with the force $2lil'i'/L^2$ when they are perpendicular to the connecting line between them, and repel each other with the force $lil'i'/L^2$ when they lie in the line. In other positions they are reduced by the parallelogram of forces to components, which either have one of the above relative positions, or are perpendicular to each other, and which therefore have no mutual action.

Magnetic Moment of a Closed Circuit.—A plane surrounded by a current i measured as above, which encloses the surface f, acts at a distance like a magnet perpendicular to the surface, of which the moment $M = fi$. We may therefore define a *unit current* as that current which, enclosing a unit area, acts at a distance as a unit magnet.

Proof for a circular current of radius r, which acts on a magnetic pole μ in its axis at the distance L. Each small portion λ exerts the force $k = \lambda i\mu/(L^2 + r^2)$. The component of this force towards the axis is $= k \,.\, r/\sqrt{L^2 + r^2} = \lambda \,.\, ri\mu/(L^2 + r^2)^{\frac{3}{2}}$. The sum of all these components is $2\pi r \,.\, ri\mu/(L^2 + r^2)^{\frac{3}{2}}$, or, for a large L, is equal to $2\pi r^2 \,.\, i\mu/L^3$. The other force-components are eliminated. The current, therefore, acts like a magnet of moment $\pi r^2 i$.

Bobbins.—A cylindrical bobbin, uniformly wound with n coils per unit of length, acts *externally* exactly as if the two end surfaces were coated with free magnetism of the surface-density ni. In the *interior* of a bobbin, which is long in proportion to its diameter, a magnetic field of the strength of $2\pi ni$ is produced at the end surfaces, and at some distance within it becomes constant $= 4\pi ni$.

The cm. g. unit of current-strength is 100 times greater than the mm. mg. unit. The current 1 cm.$^{\frac{1}{2}}$ g.$^{\frac{1}{2}}$ sec.$^{-1}$ or 1 weber passes in 1 sec. through each section 30×10^9 electrostatic cm. g. units; it decomposes in 1 sec. 0·933 mg. of water, or separates 11·18 mg. silver (Table 27), Weber's "electro-chemical equivalent."

Technical Unit, 1 ampere = 0·1 cm.$^{\frac{1}{2}}$ g.$^{\frac{1}{2}}$ sec.$^{-1}$. *—The quantity of electricity of 1 ampere-sec. corresponds to 30×10^8 electrostatic units or 0·0933 mg. water or 1·118 mg. silver.

(19*a*.) *Current-Quantity, Electrical Quantity (Electromagnetic*

* This unit has the disadvantage that current-strengths obtained from direct measurements with the tangent compass, etc., must be multiplied by 10 to express them in it, and that it cannot be used direct to calculate the magnetic action of a current, *e.g.* the magnetic field within a bobbin, but must be first reduced to absolute cm. g. measure by division by 10, in order to calculate further with cm. g. If we calculate, as is customary technically with ampere windings per cm. of a bobbin, the result is the same as if we had adhered to the mm. mg. system.

It would have been much better at once to have introduced the current-strength 1 cm. g. as technical unit, since very powerful currents are often practically employed. (Perhaps such a unit might provisionally be called a "decampere."—*Trans.*)

Measure) $\epsilon = [l^{\frac{1}{2}} m^{\frac{1}{2}}]$.—The quantity passed through a section of the circuit in unit time by unit current forms the unit quantity in this measure.

Technical Unit.—The quantity which passes through section of circuit in 1 sec. with 1 ampere current or 1 ampere-sec. is called *1 coulomb* $= 0{\cdot}1$ cm.$^{\frac{1}{2}}$ g.$^{\frac{1}{2}}$, and is 30×10^8 electrostatic units. It separates 1·118 mg. silver (19).

(20.) *Electromotive Force or Potential Difference* (*Electromagnetic Measure*) $e = [l^{\frac{3}{2}} m^{\frac{1}{2}} t^{-2}]$.—The absolute measure for this magnitude is deduced by Weber from the phenomena of magneto-induction. The law may be stated in its simplest case as follows:—In a field of uniform magnetic intensity H (16) a rectilinear conductor of length l is moved perpendicularly to itself and to the direction of H with a velocity u. By this motion an electromotive force e is induced in the conductor proportional to the length l, the magnetic intensity H, and the velocity u. Taking simply $e = lHu$, we have as *unit of electromotive force* that force which is induced in a rectilinear conductor of unit-length, moving with unit velocity across a unit magnetic field in a direction at right angles to itself and to the magnetic force.

The electromotive force is given above as length × magnetic intensity × velocity $= l \times l^{-\frac{1}{2}} m^{\frac{1}{2}} t^{-1} \times lt^{-1} = l^{\frac{3}{2}} m^{\frac{1}{2}} t^{-2}$.

If, for instance, in Central Germany, where the total magnetic intensity $= 0{\cdot}45$ cm.$^{-1}$ g.$^{\frac{1}{2}}$ sec.$^{-1}$, we hold a straight wire of 1$^{\text{m.}}$ length perpendicular to the line of dip, and move it perpendicular to itself, and to the magnetic dip, with a velocity of 1 m. per second, the induced electromotive force $= 100 \times 0{\cdot}45 \times 100 = 4500$ cm.$^{\frac{3}{2}}$ g.$^{\frac{1}{2}}$ sec.$^{-2}$.

Law of Magnetic Induction after Neumann.—The same absolute unit of electromotive force in the following form forms the basis of the law of induction. A conducting wire of any form is moved in the vicinity of magnets with the velocity u. To obtain the *E.M.F.* induced in the conductor, it may be figured as traversed by a current of 1 weber. Forces will then be exerted by the magnets on the unit current, of which p is the sum of the components at any instant in the direction of the actual motion. The induced

E.M.F. at this instant is then $e = -pu$. In the case of rotation p is the component of the moment of rotation in the plane of rotation, and u the angular velocity.

Lines of Force.—In many cases the law of induction is conveniently put in the following way. If a conductor be moved in a magnetic field (or a magnet in the neighbourhood of a conductor), the *E.M.F.* is equal to the number of lines of force cut by the conductor in unit of time (comp. 16), regard being had to their sign.

The *E.M.F.* 1 cm.$^{\frac{3}{2}}$ g.$^{\frac{1}{2}}$ sec.$^{-2}$ is 1000 mm.$^{\frac{3}{2}}$ mg.$^{\frac{1}{2}}$ sec.$^{-2}$ = $\frac{1}{30}10^9$ electrostatic cm. g. units of potential, or about $\frac{1}{11}10^7$ Daniell or $\frac{1}{19}10^7$ Bunsen elements.

Technical Unit.—1 volt $= 10^8$ cm.$^{\frac{3}{2}}$ g.$^{\frac{1}{2}}$ sec.$^{-2}$. 1 electrostatic cm. g. unit of potential $= 300$ volts. 1 Daniell element = about 1·1 – 1·2 volts. 1 Bunsen = about 1·9 volts.

Induction by Earth-Inductor (**80, 82**) is given in absolute measure by the following considerations.

We figure the coils as projected on a plane perpendicular to the direction of the earth's magnetism. Let the sum of the surfaces surrounded by all the coils change its amount at a certain instant during the revolution by the small magnitude df in the short time dt. At this instant, therefore, the induced electromotive force in absolute measure = the magnetic intensity H multiplied by the velocity $\frac{df}{dt}$ of the change of plane; and $e = H\frac{df}{dt}$.

If the inductor be turned through 180° from an original position perpendicular to the direction of H, the integral value of this induction impulse is $\int e dt = 2fH$.

This rule is included in the following more general statement. A closed conductor of the surface of coils f is moved in a magnetic field, which is not necessarily homogeneous. H_1 and H_2 are the components of magnetic intensity perpendicular to the coil surfaces at the beginning and end of the motion (+ or –). Then the integral value of the induced *E.M.F.* is $\int e dt = f(H_1 - H_2)$. If an inductor perpendicular to a field of intensity H is completely withdrawn from it, $\int e dt = fH$.

Magneto-Inductor (**81**).—A short magnet of moment M, from a great distance, is pushed into the middle of the axis of a bobbin which is long in proportion to its diameter, or similarly withdrawn. The integral value of the *E.M.F.* so induced is $4\pi nM$, where n is the number of coils per unit length of the bobbin axis.

Potential Difference.—The *E.M.F.* of a battery is proportional to the *P.D.* of the poles when the circuit is open. Considering the two magnitudes as identical, we obtain the conception of potential in the electromagnetic system also, which may be defined as the magnitude of which the diminution, or negative differential coefficient, gives the force exerted on unit quantity of electricity.

Formerly it was usual to express resistances in Siemens units, and current-strengths in mm. mg. units, and following Ohm's law, the unit of *E.M.F.* was expressed as Siem. units $\times$ mm.$^{\frac{1}{2}}$ mg.$^{\frac{1}{2}}$ sec.$^{-1}$, which is $=0{\cdot}094$ volts.

(20*a*.) *Capacity (Electromagnetic Measure)* $c=[l^{-1}t^2]$.—The *Unit of Capacity* in electromagnetic measure is that of a condenser which, charged by the *E.M.F.*, or to the potential 1 cm.$^{\frac{3}{2}}$ g.$^{\frac{1}{2}}$ sec.$^{-2}$ (20) contains the quantity of electricity 1 cm.$^{\frac{1}{2}}$ g.$^{\frac{1}{2}}$ (19*a*).

Since in the electrostatic cm. g. system the unit of el. quantity is 3×10^{10} times smaller, and that of potential is 3×10^{10} times greater than in the electromagnetic system, the unit of capacity of the latter is 9×10^{20} times smaller than the electrostatic.

Technical Unit.—The capacity of a condenser which contains the quantity 1 coulomb or ampere-sec. when charged to the potential 1 volt is 1 farad $=10^{-9}$ cm.$^{-1}$ sec.^{2} or $=9 \times 10^{11}$ electrostatic units of capacity (comp. 13). The microfarad is $\frac{1}{1000000}$ part of the farad.

An air condenser of surface f cm.^{2} and (small) separation l cm. has a capacity $f/4\pi l$ cm. electrostatic units (13 and 13*a*) or $f/(4\pi l \times 9 \times 10^5)$ microfarads. For $f=100$ cm.^{2} and $l=0{\cdot}1$ cm., for instance, the capacity is $100/(4 \times 3{\cdot}14 \times 0{\cdot}1 \times 9 \times 10^5)=$ $0{\cdot}000088$ microfarads.

(20*b*.) *Coefficient of Self-induction or Electrodynamic Self-Potential of a Conductor*, $\Pi=[l]$.—This is the factor with which the rate of change di/dt of a current must be multiplied to obtain the opposing *E.M.F.* e induced in the conductor; that is $e=\Pi\,\dfrac{di}{dt}$.

The el. dynamic potential of two closed conductors on each other is $\int\int\frac{1}{r}\cos(dl_1, dl_2) \times dl_1, dl_2$, where dl_1 and dl_2 are the elements of length of the conductors at angles measured in a given direction, and r the distance between dl_1 and dl_2 (Neumann).

(21.) *Resistance to Conduction* (*Electromagnetic Measure*), $w = [lt^{-1}]$.—In the absolute (Weber's) system of measurement we make use of Ohm's law to obtain a unit of resistance from the units of current and electromotive force, and take *as unit the resistance of a conductor,* in which unit electromotive force produces a unit current. Dimensions $= lt^{-1}$.

$$\text{For resistance} = \frac{\text{electromotive force}}{\text{current}} = \frac{l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-2}}{l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}} = lt^{-1}$$

A current-strength may therefore be expressed as a velocity, and may actually be so physically conceived. For instance, the resistance of a straight wire of unit-length is that velocity with which it must move through a unit magnetic field, under the normal conditions (p. 420), in order to produce in it a unit current, its ends being connected by a conductor without resistance, and which does not experience induction.

1 cm.^{3} cube of mercury at 0° C. has the resistance 94080 cm./sec.

1 electrostatic unit cm.$^{-1}$ sec. $= 900 \cdot 10^{18}$ cm./sec. electromagnetic units.

Technical Unit.—1 ohm $= 10^9$ cm./sec. $= 1$ volt/ampere, $= 1{\cdot}063$ Siemens's units or m./mm.^{2} Hg. at 0° C. $= \frac{1}{900}\ 10^{-9}$ electrostatic cm. g. sec. units of resistance.

"*Legal*" *System.*—1 legal ohm (German) $= 1{\cdot}060$ m./mm.^{2} Hg. at 0° C. Probably in the future the legal definitions will be 1 ohm $= 1{\cdot}063$ m./mm.^{2} Hg. at 0°; 1 ampere $=$ $1{\cdot}118$ mg. Ag. /sec.; 1 volt $=$ ohm $\times$ ampere; 1 watt $=$ volt $\times$ ampere. At present the legal ohm and volt are probably $1{\cdot}063/1{\cdot}060 = 1{\cdot}0028$ too small.

Specific Resistance $[l^2t^{-1}]$. — A conductor has *unit specific resistance,* which in a column of unit length and section possesses unit resistance.

In the electromagnetic cm. g. sec. system the specific resistance of mercury (that is, the resistance of a 1 cm. cube of Hg. at 0° C.)

is 94080 cm./sec. If, however, the resistance is measured in ohms, the length in m. and the section in mm.^{2}, the specific resistance is 0·9408; or with the "legal" ohm 0·9437.

(22.) *Current-Work, Current-Heat.*—The internal work of the current, which manifests itself in the warming of a conductor, is proportional to the sq. of the current-strength × resistance × time of action; or, what means the same, to the *E.M.F.* × current-strength × time (Joule).

The advantage of the absolute system is again evident in the fact that by its employment the above proportionality becomes equality, and the current-work L may be expressed as

$$L = i^2wt = eit.$$

This statement is true not only in the electromagnetic, but in the electrostatic system, as it is easily seen that in both cases the product of electromotive force (potential) × current-strength × work has the dimensions l^2mt^{-2}, which is that of *work*. If we adopt as unit of heat the quantity which is equivalent to unit of work, then L is the current heat (Clausius, Thomson).

Technical Unit of Current-Action, i.e. *Current-Work per sec.*—
1 watt = 1 volt × 1 amp. (see also above).

The above statement requires no proof for the electrostatic system. For the electromagnetic, it follows from the law of magnetic induction in a moving conductor, as expressed on p. 420, in connection with that of the conservation of energy. In a closed conductor, which is moved under the influence of a magnet, an induced current is produced, which exerts a mechanical ("ponderomotive") force on the magnet, which is always opposed to that causing the actual motion. By this motion, therefore, work is done which is equal to the product of the resisting force and the distance passed over. The distance is ut, where u = the velocity, and t the duration of the motion; the force is always proportional to i, the strength of the induced current. We may take the force as pi, and hence have $piut$ = the work performed.

p obviously signifies that force which will be exerted by a unit-current in the conductor on the magnet under the given

conditions. But as the law of induction (p. 420) asserts that pu is the electromotive force e in absolute measure, we have also for the work performed, $piut = eit$. If, therefore, we move a conductor under the influence of magnetic forces, and under such conditions that the *E.M.F.* e and the current i arise by magnetic induction, we perform in the time t the mechanical work eit or i^2wt.

Now, since in a metallic conductor the heat produced by the current is the only result of this work (in electrolysis the chemical action would also come into consideration), it follows from the law of the equivalence of work and heat that eit or i^2wt stands for that quantity of heat into which the mechanical work is converted by means of the current; and that quantity of heat is naturally taken as unit which is equivalent to unit of work.

But necessarily the heat liberated in the conductor is due to the interior action of the current, and hence we have in i^2wt or eit the amount of heat liberated by a current i when it traverses a conductor of resistance w, or is produced by the electromotive force e; or, in other words, its interior work.

Take, for instance, the current 1 cm.$^{\frac{1}{2}}$ g.$^{\frac{1}{2}}$ sec.$^{-1}$ in a conductor of resistance of 1 ohm $= 10^9$ cm. sec.$^{-1}$. In this case the work per sec. $= 10^9$ cm.^{2} g. sec.$^{-2}$. Now, since 42000000 of such units correspond to 1 water-g. calorie (see No. 7), the current 1 [cm. g.] in the resistance 1 ohm develops heat $= 10^9/42000000 = 24$ g. calories. According to the expression $L = i^2wt$, and since also 1 amp. $= 0{\cdot}1$ [cm. g.], the current i over w ohms resistance develops in t secs. $0{\cdot}24\ i^2wt$ g. calories of heat.

We may also say an *E.M.F.* of 1 volt $= 10^8$ cm.$^{\frac{3}{2}}$ g.$^{\frac{1}{2}}$ sec.$^{-2}$ in producing a current of 1 amp. $= 0{\cdot}1$ cm.$^{\frac{1}{2}}$ g.$^{\frac{1}{2}}$ sec.$^{-1}$ performs the work of 1 volt-amp.-sec. $= 10^7$ cm.^{2} g. sec.$^{-2}$. If we wish to convert this into kilogrammeters, we find (No. 7) that 1 kg.-wt. $\times$ meter $= 98060000$ cm.^{2} g. sec.$^{-2}$. By division, the work 1 volt-amp.-sec. $= 0{\cdot}102$ kg.-wt. $\times$ meter. Reckoned into heat, this gives, as before, $102/428 = 0{\cdot}24$ g. calories.

If we reckon 1 horse-power $= 75$ kg.-wt. $\times$ meters, then 1 volt $\times$ ampere $= 1$ watt $= 0{\cdot}102$ kg.-wt. $\times$ meter/sec. $= 0{\cdot}00136$ horse-power.

We may now define Weber's units in relation to the unit of current in the following manner:—*The unit of electromotive force* is that force which, in producing a unit current, performs unit work in unit time.

Or the *unit of resistance* is the resistance of that conductor in which unit current performs unit work in unit time.

TABLES

TABLE 1.—Density of some Bodies.

Aluminium	2·6	Cast Steel	7·8	Acetic Acid at 15°	1·053
Bismuth	9·8	Ivory	1·9	Alcohol ,,	0·7937
Brass	8·1–8·6	Lead	11·3	Amyl Alcohol ,,	0·809
Bronze	8·7	Nickel	8·9	Anilin ,,	1·023
Calcspar	2·71	Platinum	21·5	Benzol ,,	0·884
Copper	8·5–8·9	Quartz	2·65	Carbon Bisul-	
Cork	0·2	Silver	10·4	phide ,,	1·270
German Silver	8·5	Sulphur	2·0	Chloroform ,,	1·499
Glass	2·4–2·6	Tin	7·3	Ether ,,	0·720
Flint Glass	3·0–5·9	Wax	0·96	Formic Acid ,,	1·21
Gold	19·3	Wood, Ebony	1·2	Glycerine ,,	1·260
Ice	0·9167	,, Beech	0·7	Nitrobenzol ,,	1·20
Iron, Wrought	7·8	,, Oak	0·7	Olive Oil ,,	0·915
,, Cast	7·1–7·6	,, Pine	0·5	Toluol ,,	0·885
,, Wire	7·7	Zinc	7·1	Turpentine ,,	0·87

Mercury . . at 0° 13·596

	At 0° Temp. and 760 mm. Pressure compared to Water.	Compared to Air at similar Pressure and Temperature.	Compared to Hydrogen.
Air	0·0012931	1·00000	14·445
Oxygen	0·0014291	1·1052	15·964
Nitrogen	0·0012544	0·9701	14·013
Hydrogen	0·00008952	0·06923	1·000
Carbonic Dioxide	0·001965	1·520	21·95
Mixed Gases from Electrolysis of Water	0·0005360	0·4145	5·987
Aqueous Vapour	0·000804	0·6218	8·982

TABLE 2.—Reduction of Arbitrary Hydrometer Scales.

Lighter than Water.				Heavier than Water.			
Sp. gr.	Baumé.	Beck.	Cartier.	Sp. gr.	Baumé.	Beck.	Twaddell.
	°	°	°		°	°	°
0·75	58·4	56·7	...	1·0	0·0	0·0	0·0
0·80	46·3	42·5	43·0	1·1	13·2	15·4	20·0
0·85	35·6	30·0	33·6	1·2	24·3	28·3	40·0
0·90	26·1	18·9	25·2	1·3	33·7	39·2	60·0
0·95	17·7	8·9	17·7	1·4	41·8	48·6	80·0
1·00	10·0	0·0	11·0	1·5	48·8	56·7	100·0
				1·6	54·9	63·7	120·0
				1·7	60·0	70·0	140·0
				1·8	65·0	76·0	160·0
				1·9	69·0	81·0	180·0
				2·0	73·0	85·0	200·0

TABLE 3.

SPECIFIC GRAVITY OF AQUEOUS SOLUTIONS AT 15° REFERRED TO WATER OF 4°.

Mostly from Gerlach (*Zeitsch. f. Anal. Chemie*, viii. 279, 1869) and Kohlrausch (*Pogg. Ann.* clix. 257, 1876; *Wied. Ann.* vi. 38, 1879), also from Carius, Lunge, Mendelejeff, Schiff.

The percentage signifies the weight of the substance contained in 100 parts by weight of the solution. The salts are anhydrous.

%	KHO.	KCl.	KBr.	KI.	KNO_3.	K_2SO_4.	K_2CO_3.	$K_2Cr_2O_7$.	%
0	0·999	0·9991	0·999	0·999	0·999	0·999	0·999	0·999	0
5	1·045	1·0316	1·034	1·037	1·031	1·040	1·045	1·036	5
10	1·092	1·0649	1·071	1·077	1·064	(1·083)	1·092	1·072	10
15	1·141	1·0994	1·112	1·119	1·099	...	1·141	1·109	15
20	1·191	1·1351	1·155	1·165	1·135	...	1·192	...	20
25	1·242	(1·172)	1·203	1·217	...	...	1·245	...	25
30	1·295	...	1·253	1·270	...	...	1·300	...	30
35	1·349	...	1·308	1·330	...	...	1·358	...	35
40	1·406	...	1·367	1·395	...	...	1·417	...	40
45	1·466	...	1·431	1·468	...	...	1·479	...	45
50	1·528	...	1·499	1·545	...	...	1·543	...	50
55	...	...	...	1·635	...	...	...	...	55
60	...	...	...	1·733	...	...	...	...	60

%	NH_3.	NH_4Cl.	NaHO.	NaCl.	$NaNO_3$.	$Na\bar{A}$.	Na_2SO_4.	Na_2CO_3.	%
0	0·999	0·9991	0·999	0·999	0·999	0·999	0·999	0·999	0
5	0·978	1·0149	1·056	1·035	1·032	1·026	1·045	1·052	5
10	0·958	1·0299	1·111	1·072	1·067	1·052	1·092	1·105	10
15	0·941	1·0443	1·166	1·110	1·103	1·078	1·143	(1·159)	15
20	0·924	1·0584	1·222	1·150	1·141	1·105	...	...	20
25	0·910	1·0721	1·277	1·191	1·181	1·133	...	...	25
30	0·897	...	1·333	...	1·223	1·161	...	...	30
35	0·885	...	1·387	...	1·267	...	...	...	35
40	...	...	1·442	...	1·314	...	...	...	40
45	...	...	1·496	...	1·365	...	...	...	45
50	...	...	1·548	...	1·417	...	...	...	50

TABLE 3—*continued.*

%	LiCl.	$BaCl_2$.	$SrCl_2$.	$CaCl_2$.	$MgCl_2$.	$MgSO_4$.	$ZnSO_4$.	$CuSO_4$.	%
0	0·999	0·999	0·999	0·999	0·999	0·9991	0·999	0·999	0
5	1·029	1·045	1·044	1·042	1·041	1·0507	1·052	1·050	5
10	1·057	1·094	1·092	1·086	1·085	1·1044	1·108	1·103	10
15	1·085	1·148	1·143	1·133	1·130	1·1612	1·168	1·161	15
20	1·116	1·205	1·198	1·181	1·177	1·2211	1·236	(1·225)	20
25	1·147	1·269	1·257	1·232	1·226	1·2837	1·307	...	25
30	1·181	...	1·321	1·286	1·278	...	1·382	...	30
35	1·217	...	...	1·343	1·333	...	...	...	35
40	1·255	...	...	1·402	...	...	...	...	40

%	$AgNO_3$.	$Pb\bar{A}_2$.	HCl.	HNO_3.	H_2SO_4.	H_3PO_4.	Alcohol.	Sugar at 17·5°.	%
0	0·999	0·999	0·9991	0·999	0·9991	0·999	0·9991	0·9987	0
5	1·043	1·037	1·0242	1·029	1·0334	1·027	0·9904	1·0184	5
10	1·090	1·076	1·0490	1·058	1·0687	1·055	0·9831	1·0388	10
15	1·141	1·119	1·0744	1·089	1·1048	1·084	0·9769	1·0600	15
20	1·197	1·164	1·1001	1·121	1·1430	1·115	0·9708	1·0819	20
25	1·257	1·213	1·1262	1·154	1·1816	1·147	0·9644	1·1047	25
30	1·323	1·266	1·1524	1·187	1·223	1·181	0·9569	1·1282	30
35	1·396	1·324	1·1775	1·220	1·264	1·216	0·9485	1·1526	35
40	1·479	1·388	1·2007	1·253	1·307	1·253	0·9390	1·1780	40
45	1·572	...	...	1·287	1·352	1·292	0·9287	1·2041	45
50	1·677	...	...	1·320	1·399	1·333	0·9179	1·2313	50
55	1·792	...	...	1·350	1·449	1·376	0·9068	1·2593	55
60	1·919	...	...	1·377	1·503	1·421	0·8954	1·2883	60
65	...	...	...	1·402	1·559	1·467	0·8838	1·3183	65
70	...	...	...	1·424	1·616	1·515	0·8720	1·3494	70
75	...	...	...	1·443	1·675	1·566	0·8601	1·3813	75
80	...	...	...	1·461	1·733	1·619	0·8479	...	80
85	...	...	...	1·479	1·785	1·676	0·8354	...	85
90	...	...	...	1·497	1·819	1·700	0·8224	...	90
95	...	...	...	1·514	1·839	...	0·8086	...	95
100	...	...	...	1·530	1·8384	...	0·7937	...	100

TABLE 3A.

Aqueous Normal Solutions at 18°; Contents, Density, Electrical Conductivity, and Transference of Ions.

A Equivalent weight (O = 16·00) equal to the contents grm/lit.
p Percentage in 100 parts by weight.
s_{18} Specific gravity of the solution.
Δs_{18} Decrease of s for 1° at 18° (Gerlach).
k_{18} Electrical conductivity, referred to Mercury 0°.
$\Delta k/k_{18}$ Relative increase of k for 1° at nearly 18°.
n Electrical transference number of the Anion (Hittorf, Kuschel).

s, k, and Δk from F. Kohlrausch, *Wied. Ann.* vi. 148, 1879; xxvi. 174 and 195, 1885.

	A	p	s_{18}	Δs_{18}	$10^8.k_{18}$	$\Delta k/k_{18}$	n
KOH	56·14	5·357	1·0479	0·00020	1719	0·0186	0·74
KCl	74·59	7·138	1·0450	0·00028	917	0·0193	0·51
KBr	119·1	11·01	1·0814	...	960	0·0190	0·52
KI	166·04	14·845	1·1185	...	970	0·0190	0·51
KNO_3	101·18	9·543	1·0602	...	752	0·0200	0·49
$KC_2H_3O_2$	98·14	9·375	1·0468	...	594	0·0215	0·33
$\frac{1}{2}K_2SO_4$	87·17	8·177	1·0660	0·00026	672	0·0205	0·50
$\frac{1}{2}K_2CO_3$	69·14	6·537	1·0577	0·00027	662	0·0215	0·43
NH_4Cl	53·49	5·268	1·0153	0·00024	906	0·0194	0·51
$NaOH$	40·06	3·844	1·0422	0·00022	1490	0·0197	0·82
$NaCl$	58·51	5·630	1·0392	0·00028	696	0·0212	0·63
$NaNO_3$	85·10	8·071	1·0544	...	617	0·0215	0·61
$NaC_2H_3O_2$	82·06	7·898	1·040	0·00022	386	0·0250	0·42
$\frac{1}{2}Na_2SO_4$	71·09	6·704	1·0604	0·00031	477	0·0236	0·64
$\frac{1}{2}Na_2CO_3$	53·06	5·045	1·0517	0·00029	426	0·0246	0·55
$LiOH$	24·03	2·343	1·0258	...	1253	0·0196	0·88
$LiCl$	42·48	4·157	1·0228	0·00022	591	0·0220	0·74
$\frac{1}{2}Li_2SO_4$	55·06	5·271	1·0446	...	387	0·0231	(0·7)
$\frac{1}{2}BaCl_2$	104·0	9·550	1·0890	0·00031	658	0·0202	0·64
$\frac{1}{2}SrCl_2$	79·20	7·420	1·0674	0·00028	640	0·0207	0·65
$\frac{1}{2}CaCl_2$	55·45	5·313	1·0436	0·00025	633	0·0207	0·68
$\frac{1}{2}MgCl_2$	47·63	4·589	1·0379	0·00023	593	0·0217	0·71
$\frac{1}{2}MgSO_4$	60·22	5·695	1·0574	0·00027	271	0·0225	0·66
$\frac{1}{2}ZnCl_2$	68·1	6·435	1·0583	...	514	0·0022	(0·7)
$\frac{1}{2}ZnSO_4$	80·7	7·480	1·0789	...	248	0·0022	0·68
$\frac{1}{2}CuSO_4$	79·7	7·397	1·0775	...	241	0·0022	0·70
$AgNO_3$	170·0	14·91	1·140	...	634	0·0210	0·50
HCl	36·45	3·587	1·0162	...	2780	0·0159	0·17
HNO_3	63·04	6·109	1·0319	...	2770	0·0150	0·17
$\frac{1}{2}H_2SO_4$	49·06	4·762	1·0302	0·00024	1820	0·0120	0·17

TABLE 4.

Density Q of Water at Temperature $t°$.

(From determinations of Despretz, Hagen, Hallström, Jolly, Kopp, Matthiessen, Pierre, and Rosetti.)

Also Volume V of a glass vessel at 15°, which at the temperature in the table appears to contain 1 grm. of water when weighed against brass weights in air of density 0·00120 (compare p. 71).

t.	Q.	Diff.	V.	Diff.
0°	0·99988	5	1·00155	− 7
1	0·99993	4	1·00148	− 6
2	0·99997	2	1·00142	− 5
3	0·99999	1	1·00137	− 3
4	1·00000	1	1·00134	− 2
5	0·99999	2	1·00132	
6	0·99997	4	1·00132	+ 1
7	0·99993	5	1·00133	+ 2
8	0·99988	6	1·00135	+ 4
9	0·99982	8	1·00139	+ 5
10	0·99974	9	1·00144	+ 7
11	0·99965	10	1·00151	+ 8
12	0·99955	12	1·00159	+ 9
13	0·99943	13	1·00168	+ 11
14	0·99930	15	1·00179	+ 12
15	0·99915	15	1·00191	+ 13
16	0·99900	16	1·00204	+ 14
17	0·99884	18	1·00218	+ 15
18	0·99866	19	1·00233	+ 16
19	0·99847	20	1·00249	+ 18
20	0·99827	21	1·00267	+ 19
21	0·99806	22	1·00286	+ 19
22	0·99784	23	1·00305	+ 21
23	0·99761	23	1·00326	+ 21
24	0·99738	25	1·00347	+ 22
25	0·99713	25	1·00369	+ 23
[illegible]	0·99688	27	1·00392	+ 24
27	0·99661	27	1·00416	+ 25
28	0·99634	28	1·00441	+ 26
29	0·99606	29	1·00467	+ 27
30	0·99577		1·00494	

TABLE 5.

Specific Volume of Water.

Volume of 1 Grm. of Water in Cubic Centimeters between 0° and 100°.

Temp.	Volume.	Increase per 1°.
0°	1·0001	
4	1·0000	
10	1·0003	0·00012
15	1·0009	0·00016
20	1·0017	0·00024
25	1·0029	0·00028
30	1·0043	0·00032
[illegible]	1·0059	0·00036
40	1·0077	0·00040
45	1·0097	0·00046
50	1·0120	0·00048
55	1·0144	0·00050
60	1·0169	0·00056
65	1·0197	0·00058
70	1·0226	0·00062
75	1·0257	0·00064
80	1·0289	0·00066
85	1·0322	0·00070
90	1·0357	0·00074
95	1·0394	0·00076
100	1·0432	

TABLE 6.

Density of Dry Atmospheric Air.

Compared with Water at 4° C.

For Temperature t and Barometric Pressure H (in Lat. 45°) (Comp. **15**.)

t.	H = 700mm.	710mm.	720mm.	730mm.	740mm.	750mm.	760mm.	770mm.	Prop. Parts.	
	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00		
0°	1191	1208	1225	1242	1259	1276	1293	1310	17	
1	1187	1204	1221	1237	1254	1271	1288	1305	1mm.	·2
2	1182	1199	1216	1233	1250	1267	1284	1301	2	3
3	1178	1195	1212	1228	1245	1262	1279	1296	3	5
4	1174	1191	1207	1224	1241	1258	1274	1291	4	7
5	1170	1186	1203	1220	1236	1253	1270	1286	5	8
6	1165	1182	1199	1215	1232	1249	1265	1282	6	10
7	1161	1178	1194	1211	1227	1244	1261	1277	7	12
8	1157	1174	1190	1207	1223	1240	1256	1273	8	14
9	1153	1169	1186	1202	1219	1235	1252	1268	9	15
10°	1149	1165	1181	1198	1214	1231	1247	1264	16	
11	1145	1161	1177	1194	1210	1227	1243	1259	1mm.	2
12	1141	1157	1173	1190	1206	1222	1238	1255	2	3
13	1137	1153	1169	1185	1202	1218	1234	1250	3	5
14	1133	1149	1165	1181	1198	1214	1230	1246	4	6
15	1129	1145	1161	1177	1193	1209	1226	1242	5	8
16	1125	1141	1157	1173	1189	1205	1221	1237	6	10
17	1121	1137	1153	1169	1185	1201	1217	1233	7	11
18	1117	1133	1149	1165	1181	1197	1213	1229	8	13
19	1113	1129	1145	1161	1177	1193	1209	1225	9	14
20°	1110	1125	1141	1157	1173	1189	1205	1220	15	
21	1106	1122	1137	1153	1169	1185	1200	1216	1mm.	1
22	1102	1118	1133	1149	1165	1181	1196	1212	2	3
23	1098	1114	1130	1145	1161	1177	1192	1208	3	4
24	1095	1110	1126	1141	1157	1173	1188	1204	4	6
25	1091	1106	1122	1138	1153	1169	1184	1200	5	7
26	1087	1103	1118	1134	1149	1165	1180	1196	6	9
27	1084	1099	1115	1130	1146	1161	1176	1192	7	10
28	1080	1095	1111	1126	1142	1157	1173	1188	8	12
29	1076	1092	1107	1123	1138	1153	1169	1184	9	13
30°	1073	1088	1103	1119	1134	1149	1165	1180		

TABLE 7.

Reduction of Volume of Gas to 0° C. and 760 mm.

t.	$1+at$.	t.	$1+at$.	t.	$1+at$.	H.	$\frac{H.}{760}$.	H.	$\frac{H.}{760}$.
0°	1·0000	41°	1·1505	82°	1·3009	mm.		mm.	
1	1·0037	42	1·1541	83	1·3046	700	0·9211	740	0·9737
2	1·0073	43	1·1578	84	1·3083	701	0·9224	741	0·9750
3	1·0110	44	1·1615	85	1·3119	702	0·9237	742	0·9763
4	1·0147	45	1·1651	86	1·3156	703	0·9250	743	0·9776
5	1·0183	46	1·1688	87	1·3193	704	0·9263	744	0·9789
6	1·0220	47	1·1725	88	1·3230	705	0·9276	745	0·9803
7	1·0257	48	1·1762	89	1·3266	706	0·9289	746	0·9816
8	1·0294	49	1·1798	90	1·3303	707	0·9303	747	0·9829
9	1·0330	50	1·1835	91	1·3340	708	0·9316	748	0·9842
10	1·0367	51	1·1872	92	1·3376	709	0·9329	749	0·9855
11	1·0404	52	1·1908	93	1·3413	710	0·9342	750	0·9868
12	1·0440	53	1·1945	94	1·3450	711	0·9355	751	0·9882
13	1·0477	54	1·1982	95	1·3486	712	0·9368	752	0·9895
14	1·0514	55	1·2018	96	1·3523	713	0·9382	753	0·9908
15	1·0550	56	1·2055	97	1·3560	714	0·9395	754	0·9921
16	1·0587	57	1·2092	98	1·3597	715	0·9408	755	0·9934
17	1·0624	58	1·2129	99	1·3633	716	0·9421	756	0·9947
18	1·0661	59	1·2165	100	1·3670	717	0·9434	757	0·9961
19	1·0697	60	1·2202	101	1·3707	718	0·9447	758	0·9974
20	1·0734	61	1·2239	102	1·3743	719	0·9461	759	0·9987
21	1·0771	62	1·2275	103	1·3780	720	0·9474	760	1·0000
22	1·0807	63	1·2312	104	1·3817	721	0·9487	761	1·0013
23	1·0844	64	1·2349	105	1·3853	722	0·9500	762	1·0026
24	1·0881	65	1·2385	106	1·3890	723	0·9513	763	1·0039
25	1·0917	66	1·2422	107	1·3927	724	0·9526	764	1·0053
26	1·0954	67	1·2459	108	1·3964	725	0·9539	765	1·0066
27	1·0991	68	1·2496	109	1·4000	726	0·9553	766	1·0079
28	1·1028	69	1·2532	110	1·4037	727	0·9566	767	1·0092
29	1·1064	70	1·2569	111	1·4074	728	0·9579	768	1·0105
30	1·1101	71	1·2606	112	1·4110	729	0·9592	769	1·0118
31	1·1138	72	1·2642	113	1·4147	730	0·9605	770	1·0132
32	1·1174	73	1·2679	114	1·4184	731	0·9618	771	1·0145
33	1·1211	74	1·2716	115	1·4220	732	0·9632	772	1·0158
34	1·1248	75	1·2752	116	1·4257	733	0·9645	773	1·0171
35	1·1284	76	1·2789	117	1·4294	734	0·9658	774	1·0184
36	1·1321	77	1·2826	118	1·4331	735	0·9671	775	1·0197
37	1·1358	78	1·2863	119	1·4367	736	0·9684	776	1·0211
38	1·1395	79	1·2899	120	1·4404	737	0·9697	777	1·0224
39	1·1431	80	1·2936			738	0·9711	778	1·0237
40	1·1468	81	1·2973			739	0·9724	779	1·0250
41	1·1505	82	1·3009			740	0·9737	780	1·0263

TABLE 8.—Reduction of a Weighing with Brass Weights to Weight in vacuo.

s.	k.	s.	k.	s.	k.
0·7	+1·57	2·0	+0·458	8	+0·007
0·8	1·36	2·5	0·337	9	−0·009
0·9	1·19	3·0	0·257	10	−0·023
1·0	1·06	3·5	0·200	11	−0·034
1·1	0·95	4·0	0·157	12	−0·043
1·2	0·86	4·5	0·124	13	−0·050
1·3	0·78	5·0	0·097	14	−0·057
1·4	0·71	5·5	0·075	15	−0·063
1·5	0·66	6·0	0·057	16	−0·068
1·6	0·61	6·5	0·042	17	−0·072
1·7	0·56	7·0	0·029	18	−0·076
1·8	0·52	7·5	0·017	19	−0·080
1·9	0·49	8·0	+0·007	20	−0·083
2·0	+ 0·46			21	−0·086

$$k=1{\cdot}20\left(\frac{1}{s}-\frac{1}{8{\cdot}4}\right)$$

If the body weighed has the density s and weight in air m *grams*, mk mgrms. must be added to reduce the weight to *vacuo*. (*Cf.* **11**, II.)

TABLE 8A.—Gravity g and Length l of the Seconds Pendulum in Latitude ϕ (20, 3).

In latitude 45°, $g_{45}=980{\cdot}62$ cm/sec².

ϕ	0°	10	20	30	40	50	60	70	80	90
g	978·1	978·2	978·7	979·3	980·2	981·1	981·9	982·6	983·0	983·2 cm/sec²
$\frac{g}{g_{45}}$	0·9974	0·9976	0·9980	0·9987	0·9995	1·0005	1·0013	1·0020	1·0024	1·0026
l	99·10	99·12	99·15	99·23	99·31	99·40	99·49	99·56	99·60	99·62 cm.

TABLE 9.—Coefficients of Expansion for 1° C.

The length L of a body is increased by βL for each degree of increased temperature, and its volume V by $3\beta V$. (Compare **26.**)

Aluminium	0·000023	Brass	0·000019
Lead	0·000029	German Silver	0·000018
Iron	0·000012	Platinum	0·000009
Glass	0·0000085	Platinum-iridium	0·000009
Gold	0·000015	Silver	0·000019
Copper	0·000017	Zinc	0·000029
Vulcanite	0·00008	Tin	0·000023

Wood with the grain . . 0·000003 to 0·0000010

The volume V of mercury increases 0·000181 of its volume at 0° for 1°.

At 15° a solution of p per cent of the following substances expands for 1°:—Strong spirits of wine, $0{\cdot}0003+0{\cdot}000009\,p$; sugar, $0{\cdot}00016+0{\cdot}000004\,p$; common salt, dilute sulphuric acid, $0{\cdot}00016+0{\cdot}000010\,p$.

Cf. for expansion of solutions Table 3A.

TABLE 10.—Conducting Power for Heat.

(From Angstrom, Neumann, Kirchhoff and Hanseman, H. Weber, F. Weber, Lorenz, F. Kohlrausch.)

Through a cube of 1 cm. of which two opposite sides have a difference of temperature of 1° there passes in 1 sec. a quantity of heat measured in Gram-calories as follows:—

Bismuth	0·01	Lead	0·08
Brass	0·15 to 0·30	Platinum	0·1
Copper	0·50 to 0·90	Silver	1·0
German Silver	about 0·08	Steel	0·06 to 0·14
Glass	about 0·001	Tin	0·14
Iron	0·15 to 0·18	Zinc	0·26

TABLE 10A.—Solubilities in Water.

(From the detailed table by Less in Landolt and Börnstein's tables.)

100 parts by weight of water dissolve the following parts by weight of the anhydrous salt.

At Temp.	0°	20°	100°	At Temp.	0°	20°	100°
KCl	28	35	57	$CaCl_2$	49	74	155
KI	128	144	209	$CaSO_4$	0·19	0·21	0·17
$KClO_3$	3	7	56	$MgSO_4$	27	36	74
KNO_3	13	31	250	$ZnSO_4$	43	53	95
K_2SO_4	8	11	26	$CuSO_4$	18	24	75
$K_2Cr_2O_7$	5	12	94	$NiSO_4$	29	40	
K_2CO_3	83	94	154	$AgNo_3$	120	240	900
NH_4Cl	28	37	73	$HgCl_2$	6	7	54
$NaCl$	35·5	36·0	39·6	Cane Sugar	186	203	
Na_2CO_3	7	26	47				
$LiCl$	64	80	130				
$BaCl_2$	21	36	59				

TABLE 10B.—Absorption of Gases in Water at Atmosphere Pressure.

(Mostly from Bunsen.)

1 Liter contains when saturated

At Temp.	0°	20°
Air	0·032 grm.	0·022 grm.
Oxygen	0·059 ,,	0·041 ,,
Nitrogen	0·026 ,,	0·018 ,,
Hydrogen	0·002 ,,	0·002 ,,
Chlorine	... ,,	6·8 ,,
Carbon dioxide	3·5 ,,	1·8 ,,
Sulphuretted hydrogen	6·6 ,,	4·4 ,,
Sulphur dioxide	228 ,,	113 ,,
Ammonia	800 ,,	500 ,,

TABLE 11.

Reduction of the Barometer Reading to 0°

on account of the expansion of the Mercury and of the Scale.
(Comp. 20.)

If h is the height of the column of mercury as read off, t the temperature, β the coefficient of expansion of the scale, we must, in order to obtain the reading reduced to 0°, subtract from h the amount $(0{\cdot}000181 - \beta)ht$. The table contains this correction for a brass scale with $\beta = 0{\cdot}000019$.

If the scale is engraved on the glass tube it is sufficient to increase the numbers of the Table by $0{\cdot}008t$. See the last column.

t	Observed height (h) in mm.										+0·008t
	680.	690.	700.	710.	720.	730.	740.	750.	760.	770.	
	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
1	0·11	0·11	0·11	0·12	0·12	0·12	0·12	0·12	0·12	0·12	+0·01
2	0·22	0·22	0·23	0·23	0·23	0·24	0·24	0·24	0·25	0·25	0·02
3	0·33	0·34	0·34	0·35	0·35	0·35	0·36	0·36	0·37	0·37	0·02
4	0·44	0·45	0·45	0·46	0·47	0·47	0·48	0·49	0·49	0·50	0·03
5	0·55	0·56	0·57	0·58	0·58	0·59	0·60	0·61	0·62	0·62	0·04
6	0·66	0·67	0·68	0·69	0·70	0·71	0·72	0·73	0·74	0·75	0·05
7	0·77	0·78	0·79	0·81	0·82	0·83	0·84	0·85	0·86	0·87	0·06
8	0·88	0·89	0·91	0·92	0·93	0·95	0·96	0·97	0·98	0·99	0·06
9	0·99	1·01	1·02	1·04	1·05	1·06	1·08	1·09	1·11	1·12	0·07
10	1·10	1·12	1·13	1·15	1·17	1·18	1·20	1·22	1·23	1·25	0·08
11	1·21	1·23	1·25	1·27	1·28	1·30	1·32	1·34	1·35	1·37	0·09
12	1·32	1·34	1·36	1·38	1·40	1·42	1·44	1·46	1·48	1·50	0·10
13	1·43	1·45	1·47	1·50	1·52	1·54	1·56	1·58	1·60	1·62	0·10
14	1·54	1·56	1·59	1·61	1·63	1·66	1·68	1·70	1·72	1·75	0·11
15	1·65	1·68	1·70	1·73	1·75	1·77	1·80	1·82	1·85	1·87	0·12
16	1·76	1·79	1·81	1·84	1·87	1·89	1·92	1·94	1·97	2·00	0·13
17	1·87	1·90	1·93	1·96	1·98	2·01	2·04	2·07	2·09	2·12	0·14
18	1·98	2·01	2·04	2·07	2·10	2·13	2·16	2·19	2·22	2·25	0·14
19	2·09	2·12	2·15	2·19	2·22	2·25	2·28	2·31	2·34	2·37	0·15
20	2·20	2·24	2·27	2·30	2·33	2·37	2·40	2·43	2·46	2·49	0·16
21	2·31	2·35	2·38	2·42	2·45	2·48	2·52	2·55	2·59	2·62	0·17
22	2·42	2·46	2·49	2·53	2·57	2·60	2·64	2·67	2·71	2·74	0·18
23	2·53	2·57	2·61	2·65	2·68	2·72	2·76	2·79	2·83	2·87	0·18
24	2·64	2·68	2·72	2·76	2·80	2·84	2·88	2·92	2·95	2·99	0·19
25	2·75	2·79	2·84	2·88	2·92	2·96	3·00	3·04	3·08	3·12	0·20
26	2·86	2·91	2·95	2·99	3·03	3·07	3·12	3·16	3·20	3·24	0·21
27	2·97	3·02	3·06	3·11	3·15	3·19	3·24	3·28	3·32	3·37	0·22
28	3·08	3·13	3·18	3·22	3·27	3·31	3·36	3·40	3·45	3·49	0·22
29	3·19	3·24	3·29	3·34	3·38	3·43	3·48	3·52	3·57	3·62	0·23
30	3·30	3·35	3·40	3·45	3·50	3·55	3·60	3·65	3·69	3·74	0·24

TABLE 12.

MEAN HEIGHT b OF BAROMETER AT ELEVATION H ABOVE THE SEA-LEVEL.

Temperature of Air taken at 10° C.

H.	*H.*	*b.*	*b.*	*H.*	*H.*	*b.*	*b.*
meters.	Eng. feet.	mm.	inches.	meters.	Eng. feet.	mm.	inches.
0	0	760	29·92	1000	3280	674	26·53
100	328	751	29·57	1100	3608	666	26·22
200	656	742	29·21	1200	3936	658	25·90
300	984	733	28·85	1300	4265	650	25·59
400	1312	724	28·50	1400	4592	642	25·27
500	1640	716	28·19	1500	4920	635	25·00
600	1968	707	27·83	1600	5248	627	24·68
700	2296	699	27·52	1700	5577	620	24·41
800	2624	690	27·17	1800	5905	612	24·09
900	2952	682	26·85	1900	6233	605	23·82
1000	3280	674	26·53	2000	6561	598	23·54

TABLE 12A.

REDUCTION OF MILLIMETERS TO INCHES.

mm.	inches.	mm.	inches.
100	3·93708	710	27·9532
200	7·87415	720	28·3469
300	11·81124	730	28·7406
400	15·74832	740	29·1343
500	19·68539	750	29·5280
600	23·62247	760	29·9217
700	27·55955	770	30·3155
800	31·49663	780	30·7091
900	35·43371	790	31·1029
1000	39·37079	800	31·4966

TABLE 13.

TENSION e OF AQUEOUS VAPOUR (in mm. of Mercury), AND WEIGHT f OF AQUEOUS VAPOUR (in 1 c.c. in grms.), SATURATED AT TEMPERATURE t

According to Magnus and Regnault (**28**).

t.	e.	f.	t.	e.	f.	t.	e.	t.	e.
	mm.	g/cc.		mm.	g/cc.		mm.		mm.
−10°	2·2	2·4	13	11·1	11·2	36	44·2	64	178·6
−9	2·3	2·5	14	11·9	12·0	37	46·7	65	186·9
−8	2·5	2·7	15	12·7	12·8	38	49·3	66	195·4
−7	2·7	2·9	16	13·5	13·5	39	52·1	67	204·3
−6	2·9	3·1	17	14·4	14·4	40	54·9	68	213·5
−5	3·2	3·4	18	15·3	15·2	41	57·9	69	223·0
−4	3·4	3·7	19	16·3	16·2	42	61·1	70	232·9
−3	3·7	4·0	20	17·4	17·2	43	64·4	71	243·3
−2	3·9	4·2	21	18·5	18·2	44	67·8	72	253·9
−1	4·2	4·5	22	19·6	19·2	45	71·4	73	265·0
0	4·6	4·9	23	20·9	20·4	46	75·2	74	276·4
+1	4·9	5·2	24	22·2	21·6	47	79·1	75	288·3
2	5·3	5·6	25	23·5	22·8	48	83·2	76	300·6
3	5·7	6·0	26	25·0	24·2	49	87·5	77	313·4
4	6·1	6·4	27	26·5	25·6	50	92·0	78	326·6
5	6·5	6·8	28	28·1	27·0	51	96·6	79	340·2
6	7·0	7·3	29	29·7	28·5	52	101·5	80	354·4
7	7·5	7·8	30	31·5	30·1	53	106·6	81	369·0
8	8·0	8·2	31	33·4	...	54	111·9	82	384·2
9	8·5	8·7	32	35·4	...	55	117·4	83	399·9
10	9·1	9·3	33	37·4	...	56	123·2	84	416·1
11	9·8	10·0	34	39·6	...	57	129·2	85	432·7
12	10·4	10·6	35	41·9	...	58	135·4	86	450·1
						59	141·9	87	467·9
						60	148·7	88	486·4
						61	156·7	89	505·4
						62	163·0		
						63	170·6		

TABLE 13A.

TENSION OF AQUEOUS VAPOUR

Between 90° and 101° in mm. of Mercury of 0° (Regnault).

	90°	91°	92°	93°	94°	95°	96°	97°	98°	99°	100°
	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
·0	525·5	545·8	566·7	588·3	610·6	633·7	657·4	681·9	707·1	733·2	760·0
·1	527·5	547·8	568·8	590·5	612·9	636·0	659·8	684·4	709·7	735·8	762·7
·2	529·5	549·9	571·0	592·7	615·2	638·3	662·2	686·9	712·3	738·5	765·5
·3	531·5	552·0	573·1	595·0	617·5	640·7	664·7	689·4	714·9	741·1	768·2
·4	533·5	554·1	575·3	597·2	619·8	643·1	667·1	691·9	717·4	743·8	771·0
·5	535·5	556·2	577·4	599·4	622·1	645·4	669·5	694·4	720·0	746·5	773·7
·6	537·6	558·3	579·6	601·6	624·4	647·8	672·0	696·9	722·7	749·2	776·5
·7	539·6	560·4	581·8	603·9	626·7	650·2	674·5	699·5	725·3	751·9	779·3
·8	541·7	562·5	584·0	606·1	629·0	652·6	676·9	702·0	727·9	754·6	782·1
·9	543·7	564·6	586·1	608·4	631·3	655·0	679·4	704·6	730·5	757·3	784·9

TABLE 13B.

BOILING TEMPERATURE *t* OF WATER AT BAROMETER PRESSURE *b*.

(From Regnault's observations.)

b	*t*.	*b*	*t*.	*b*	*t*.	*b*	*t*.	*b*	*t*.
mm.	°	mm.	°	mm.	°	mm.	°	mm.	°
680	96·92	700	97·72	720	98·50	740	99·26	760	100·00
81	96·96	01	·76	21	·54	41	·30	61	·04
82	97·00	02	·80	22	·57	42	·33	62	·07
83	·05	03	·84	23	·61	43	·37	63	·11
84	·09	04	·88	24	·65	44	·41	64	·15
85	·13	05	·92	25	·69	45	·44	65	·18
86	·17	06	·96	26	·73	46	·48	66	·22
87	·21	07	97·99	27	·77	47	·52	67	·26
88	·25	08	98·03	28	·80	48	·56	68	·29
89	·29	09	·07	29	·84	49	·59	69	·33
690	·32	710	·11	730	·88	750	·63	770	·36
91	·36	11	·15	31	·92	51	·67	71	·40
92	·40	12	·19	32	·96	52	·71	72	·44
93	·44	13	·23	33	98·99	53	·74	73	·47
94	·48	14	·27	34	99·03	54	·78	74	·51
95	·52	15	·31	35	·07	55	·82	75	·55
96	·56	16	·34	36	·11	56	·85	76	·58
97	·60	17	·38	37	·14	57	·89	77	·62
98	·64	18	·42	38	·18	58	·93	78	·65
699	·68	19	·46	39	·22	59	·96	79	·69
700	97·72	720	98·50	740	99·26	760	100·00	780	100·73

TABLE 14.—Tension of Mercurial Vapour.

In mm. of mercury (Regnault, Hagen, Hertz, Ramsay, and Young).

	mm.		mm.
0°	0·01	200°	18·3
20	0·02	220	34
40	0·03	240	59
60	0·06	260	97
80	0·10	280	156
100	0·3	300	244
120	0·8	320	371
140	1·9	340	548
160	4·4	360	790
180	9·2		

TABLE 15.—Capillary Depression of Mercury in a Glass Tube.

Interpolated from observations of Mendelejeff and Gutkowsky.

	Height of the Meniscus in mm.							
Diameter	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8
mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
4	0·83	1·22	1·54	1·98	2·37	...	...	...
5	0·47	0·65	0·86	1·19	1·45	1·80	...	...
6	0·27	0·41	0·56	0·78	0·98	1·21	1·43	...
7	0·18	0·28	0·40	0·53	0·67	0·82	0·97	1·13
8	...	0·20	0·29	0·38	0·46	0·56	0·65	0·77
9	...	0·15	0·21	0·28	0·33	0·40	0·46	0·52
10	...	...	0·15	0·20	0·25	0·29	0·33	0·37
11	...	...	0·10	0·14	0·18	0·21	0·24	0·27
12	...	...	0·07	0·10	0·13	0·15	0·18	0·19
13	...	...	0·04	0·07	0·10	0·12	0·13	0·14

TABLE 16.—Specific Heats.

Alcohol	(17°) . . .	0·58	Gold	15-100°	0·032
Aniline	,, . . .	0·49	Iron	,, ...	0·113
Mercury	,, . . .	0·034	Lead	,, ...	0·032
Toluol	,, . . .	0·40	Nickel	,, ...	0·11
Turpentine	,, . . .	0·43	Platinum	,, ...	0·032
Brass	15-100° . . .	0·094	Quartz	,, ...	0·191
Calcspar	,, . . .	0·208	Silver	,, ...	0·057
Copper	,, . . .	0·094	Tin	,, ...	0·056
Glass	,, . . .	0·19	Zinc	,, ...	0·094

TABLE 16A.

MELTING-POINT AND BOILING-POINT, AND THE DEPRESSION (AND RISE) Δ PRODUCED BY THE SOLUTION OF 1 GRAM-MOLECULE IN 1000 GRAMS OF THE SOLVENT.

(*Cf.* Beckmann, L. S., *Phys. Chem.* viii. 226, 1891.)

	Melting-point.	Δ	Boiling-point.	Δ
	°	°	°	°
Acetic Acid	16·5	− 3·9	118	...
Acetate of Soda . .	58	...	...	...
Alcohol	...	...	78·3	+ 1·15
Amyl Alcohol . .	...	...	137	3·2
Aniline	− 8	...	183	2·4
Benzol	+ 5·4	5·0	80	2·7
Carbon Bisulphide . .	...	...	46·8	2·4
Carbonic Anhydride .	− 57	...	− 79	...
Chloride of Potassium .	730	...	...	...
Chloride of Sodium .	800	...	...	...
Chloroform	...	...	61·1	3·7
Ether	...	...	34·9	2·1
Formic Acid	+ 8·5	2·8	103	...
Lead	326	...	...	...
Mercury	− 39·5	...	358	...
Methyl Alcohol . .	...	...	65	...
Naphthalin	80	...	214	...
Nitrobenzol	5·3	...	210	...
Phenol	18	...	186	3·0
Rose's Metal	95	...	...	...
Sulphur	115	...	448	...
Stearic Acid	69·5	...	370	...
Tin	230	...	...	...
Toluol	...	...	110	...
Water	0	1·89	100	0·52
Wood's Metal	68	...	...	...
Xylol	15	...	137	...
Zinc	410	...	...	...

TABLE 17.—Modulus of Elasticity E, Breaking Strain p, and Velocity of Sound in some Metals when stretched at 17° C. (After Wertheim. *Cf.* **33.**)

If a wire be employed of 1 sq. mm. section, E signifies the weight in kilograms which would be required to double its length; and p the weight in kgr. which would break it. It follows that the increment of length, λ, of a wire of length l and section q mm.², caused by a stretching weight of P kgr., will be $\lambda = \frac{l}{q}\frac{P}{E}$, and a wire of q mm.² will break with a strain of qp kgr.

The numbers must only be used as approximations.

	E.	p.	u.
Lead	1,800 $\frac{\text{kg.-wt.}}{\text{mm.}^2}$	2 $\frac{\text{kg.-wt.}}{\text{mm.}^2}$	1300 $\frac{\text{m.}}{\text{sec.}}$
Iron	19,000	60	5000
Steel	21,000	80	5100
Gold	8,100	27	2100
Copper . . .	12,400	40	3700
Brass	9,000	60	3200
Platinum . . .	17,000	30	2800
Silver	7,400	29	2700
Zinc	8,700	13	3500
Tin	4,000	2	2300
Glass	7,000	...	5000
Wood fibre . . .	5 to 1,200	...	3 to 4000

TABLE 18.—Pitch and Number of Vibrations per Second of Musical Notes.

	C_{-2}.	C_{-1}.	C.	c.	c_1.	c_2.	c_3.	c_4.
C	16·35	32·70	65·41	130·8	261·7	523·3	1047	2093
$C\sharp$	17·32	34·65	69·30	138·6	277·2	554·4	1109	2218
D	18·35	36·71	73·42	146·8	293·7	587·4	1175	2350
$D\sharp$	19·44	38·89	77·79	155·6	311·2	622·3	1245	2489
E	20·60	41·20	82·41	164·8	329·7	659·3	1319	2637
F	21·82	43·65	87·31	174·6	349·2	698·5	1397	2794
$F\sharp$	23·12	46·25	92·50	185·0	370·0	740·0	1480	2960
G	24·50	49·00	98·00	196·0	392·0	784·0	1568	3136
$G\sharp$	25·95	51·91	103·8	207·6	415·3	830·6	1661	3322
A	27·50	55·00	110·0	220·0	**440·0**	880·0	1760	3520
$A\sharp$	29·13	58·27	116·5	233·1	466·2	932·3	1865	3729
B	30·86	61·73	123·6	246·9	493·9	987·7	1975	3951

TABLE 19.—LINES OF THE FLAME-SPECTRA OF THE MOST IMPORTANT LIGHT METALS,

according to Bunsen and Kirchhoff's scale; the sodium line being taken as 50, and the slit having a breadth of 1 division.

The first number denotes the position of the middle of the line upon the scale, the Roman figure indicates the brightness, I being the brightest, and the third number gives the breadth of the band when it exceeds 1 scale-division, the breadth of the slit.

S signifies that the line is quite sharp and clearly defined, *s* that it is tolerably so; the remaining lines being nebulous and ill-defined.

The lines most characteristic of each body are printed in thick type.

The brightness of the lines of *Ca*, *Sr*, and *Ba* is that of a constant spectrum. If the chlorides be employed, the spectra are at first much brighter. For Fraunhofer's lines see Table 19A.

The colours of the spectrum are approximately—red to 48, yellow to 52, green to 80, blue to 120, and violet beyond.

K.	*Na.*	*Li.*	*Ca.*	*Sr.*	*Ba.*
17·5 II. *s*		32·0 I. *S*	33·1 IV. 2	29·8 III.	
			36·7 IV.	32·1 II.	
				33·8 II.	
		45·2 IV. *s*	41·7 I. 1·5	36·3 II.	35·2 IV. 2
Faint continuous spectrum from 55 to 120	50·0 I. *S*		46·8 III. 2	39·0 III.	41·5 III. 3
			49·0 III.	41·8 III.	45·6 III. 1·5
				45·8 I.	
					52·1 IV.
			52·8 IV.		56·0 III. 2
			54·9 IV.		60·8 II. *s*
			60·8 I. 1·5		66·5 III. 3
			68·0 IV. 2	105.0 III. *S*	71·4 III. 3
					76·8 III. 2
153·0 IV. *s*			135·0 IV. *S*		82·7 IV. 4
					89·3 III. 2

TABLE 19A.—WAVE-LENGTHS IN AIR FOR THE MOST IMPORTANT LINES

in the spectra of the chemical elements and the solar spectrum with their place in Bunsen and Kirchhoff's scale. The ultra-violet lines of Cd., Zn, and Al according to Mascart and Cornu.

	10^{-6} mm.	Scale Number.
Potassium α	768	17·5
A	760·4	18
a	718·6	23
B	687·0	28·2
Lithium α	670·8	32·0
C (Hydrogen α)	656·3	34
D_1 Sodium	589·62	50·0
D_2 Sodium	589·02	50·0
Thallium	534·9	68
E	527·0	71·3
b (Magnesium middle line)	517·3	76
F (Hydrogen β)	486·1	90
Strontium δ	460·8	105
Hydrogen γ	434·0	126
G	430·7	128
h (Hydrogen δ)	410·2	147
Potassium β	404·6	153
H_1	396·6	162
H_2	393·4	166

	10^{-6} mm.
Cadmium	398·6
	360·8
	346·3
	328·8
	274·5
	257·3
	231·8
	226·6
	220·6
	214·4
Zinc	213·8
	209·9
	206·2
	202·4
Aluminium	198·8
	193·1
	185·6

TABLE 20.

Indices of Refraction of some Bodies and Rotation of Quartz at 1 mm. thick.

(From Beer's *Optics*; Ketteler, *Pogg. Ann.* cxl., and Landolt and Börnstein's Tables from observations by Baden Powell, Dale and Gladstone, Fraunhofer, Grailich, Kohlrausch, Kundt, v. Lang, Mascart, Quincke, Rudberg, Schrauf, Soret and Sarasin, Stefan, Verdet, van der Willigen, Wüllner and others. *Cf.* **39** and **46**).

The index of refraction decreases at medium temperatures for 1° increase of temperature by 0·00009 in the case of Water; by 0·0008 for D and 0·0009 for H in the case of Carbon bisulphide.

For Biaxial crystals, the numbers given are, when not otherwise stated, for the mean index.

Wave-Length $\lambda.10^{-6}$ =	A	B	C	D	E	F	G	H
	760 mm.	687	656	589	527	486	431	397
Water, 17°·5	1·3291	·3306	·3314	·3332	·3353	·3374	·3407	·3436
Alcohol, 15°·0	1·3598	·3611	·3618	·3635	·3658	·3679	·3716	·3748
Carbon Bisulphide, 16°·0 .	1·6118	·6181	·6214	·6308	·6438	·6555	·6794	·7032
Oil of Cassia, 17°·5 . .	1·5858	·5924	·5958	·6053	·6194	·6340	·6652	·7009
,, ,, light . .	1·5100	·5118	·5127	·5153	·5186	·5214	·5267	·5312
Crown Glass, heavy . .	1·6097	·6117	·6126	·6152	·6185	·6213	·6265	·6308
,, ,, light . .	1·5986	·6020	·6038	·6085	·6145	·6200	·6308	·6404
Flint Glass, heavy . .	1·7350	·7405	·7434	·7515	·7623	·7723	·7922	·811
,, ,, ordinary . .	1·6500	·6530	·6545	·6585	·6635	·6679	·6762	·6833
Calcspar, extraordinary ray	1·4828	·4840	·4847	·4864	·4888	·4908	·4946	·4978
,, ordinary ray .	1·5390	·5409	·5418	·5442	·5471	·5497	·5543	·5582
Quartz, extraordinary ray .	1·5481	·5500	·5509	·5533	·5563	·5589	·5637	·5677
Selenite, mean . . .	1·518	·519	·520	·523	·525	·528	·532	...
Arragonite, mean . .	1·674	·676	·678	·682	·686	·691	·698	·705
Topaz, mean . . .	1·608	·610	·611	·614	·617	·619	·624	·627
Rock salt	1·538	·540	·541	·545	·550	·554	·562	·569
Rotation in Quartz at 20° .	12°·7	15°·7	17°·3	21°·7	27°·5	32°·8	42°·6	51°·2

Air	1·00029	Heavy Spar	1·64
Arsenious Bromide . .	1·78	Ice	1·31
Augite (Diopside) . .	1·68	Monobromnaphthalin .	1·66
Benzol	1·50	Nitre	1·50
Beryl	1·57	Phosphorus in CS_2 . .	1·97
Canada Balsam (hard) .	1·54	Rape oil	1·47
Ether	1·36	Sugar	1·56
Felspar	1·52	Tourmaline	1·65
Fluorspar	1·44	Turpentine	1·48

The three principal Indices of Refraction for "Sodium Yellow" are—

Selenite . . .	1·529	1·522	1·520
East Indian Mica .	1·600	1·594	1·561
Arragonite . .	1·686	1·682	1·530
Topaz . . .	1·621	1·614	1·612

TABLE 20A.—Colours of Newton's Rings.

Shown by a film of Air of thickness *d* in reflected and transmitted light for perpendicular rays.

(According to Quincke, *Pogg. Ann.* cxxix. 180, 1866.)

d.	Reflected.	Transmitted.
$\frac{mm}{10^6}$	**1 ORDER.**	
0	Black	White
20	Iron gray	White
48	Lavender gray	Yellowish wh.
79	Gray blue	Brownish wh.
109	Clear gray	Yellow brown
117	Greenish wh.	Brown
129	Almost pure white	Bright red
133	Yellowish wh.	Carmine red
137	Pale straw yellow	Dark red brown
140	Straw yellow	Dark violet
153	Pure yellow	Indigo
166	Bright yellow	Blue
215	Brown yellow	Gray blue
252	Red orange	Blue green
268	Warm red	Pale green
275	Deep red	Yellow green
	2 ORDER.	
282	Purple	Bright green
287	Violet	Greenish yellow
294	Indigo	Golden yellow
332	Skyblue	Orange
364	Greenish blue	Brown orange
374	Green	Bright carmine red
413	Bright green	Purple
421	Yellowish green	Violet purple
433	Greenish yellow	Violet
455	Pure yellow	Indigo
474	Orange	Dark blue
499	Bright red orange	Greenish blue
550	Dark violet red	Green

d.	Reflected.	Transmitted.
$\frac{mm}{10^6}$	**3 ORDER.**	
564	Bright blue violet	Yellowish green
575	Indigo	Impure yellow
629	Blue (greenish)	Flesh colour
667	Sea green	Brown red
688	Intense green	Violet
713	Greenish yellow	Gray blue
747	Flesh colour	Sea green
767	Carmine red	Fine green
810	Dull purple	Dull sea green
826	Violet gray	Yellowish green
	4 ORDER.	
841	Gray blue	Greenish yellow
855	Dull sea green	Yellowish gray
872	Bluish green	Grayish red
905	Fine clear green	Carmine red
963	Clear gray green	Grayish red
1003	Gray, almost white	Grayish blue
1024	Flesh colour	Green
	5 ORDER.	
1169	Dull blue green	Dull flesh colour
1334	Dull flesh colour	Dull blue green

TABLE 21.

For Reduction of Time of Oscillation to an Infinitely Small Arc.

If the observed time of oscillation of a magnet or pendulum be t, with an arc of oscillation of α degrees, kt must be subtracted from the observed value in order to reduce the time to that of an infinitely small oscillation (**53**).

α.	k.		α.	k.		α.	k.		α.	k.	
0°	0·00000	0	10°	0·00048	10	20°	0·00190	20	30°	0·00428	29
1	000	2	11	058	11	21	210	20	31	457	30
2	002	2	12	069	11	22	230	21	32	487	31
3	004	4	13	080	13	23	251	23	33	518	32
4	008	4	14	093	14	24	274	23	34	550	33
5	012	5	15	107	15	25	297	25	35	583	93
6	017	6	16	122	16	26	322	25	36	616	95
7	023	7	17	138	16	27	347	26	37	651	35
8	030	9	18	154	18	28	373	27	38	686	37
9	039	9	19	172	18	29	400	28	39	723	[illegible]
10°	0·00048		20°	0·00190		30°	0·00428		40°	0·00761	

TABLE 21a.

Reduction of Deflection e observed on a Divided Scale when the Distance from the Mirror is A Scale-divisions (**49**).

By subtracting the number in the Table the scale-reading becomes proportional to the angle of deflection. The corrections for the tangents are equal to $\frac{3}{4}$, for the sines to $\frac{3}{8}$, of the numbers given.

A.	e=50.	100.	150.	200.	250.	300.	350.	400.	450.	500.
1000	0·04	0·33	1·11	2·60	5·02	8·54	13·33	19·5	27·1	36·3
1200	0·03	0·23	0·77	1·82	3·53	6·03	9·45	13·9	19·5	26·2
1400	0·02	0·17	0·57	1·34	2·61	4·47	7·03	10·4	14·6	19·7
1600	0·02	0·13	0·44	1·03	2·00	3·44	5·43	8·0	11·3	15·4
1800	0·01	0·10	0·35	0·82	1·59	2·73	4·30	6·4	9·0	12·3
2000	0·01	0·08	0·28	0·66	1·29	2·22	3·51	5·21	7·37	10·05
2200	0·01	0·07	0·23	0·55	1·07	1·83	2·91	4·32	6·12	8·35
2400	0·01	0·06	0·19	0·46	0·90	1·54	2·45	3·64	5·16	7·05
2600	0·01	0·05	0·16	0·39	0·77	1·32	2·09	3·11	4·42	6·03
2800	0·01	0·04	0·14	0·34	0·66	1·14	1·81	2·69	3·82	5·21
3000	0·00	0·04	0·12	0·29	0·58	0·99	1·58	2·35	3·33	4·55
3200	0·00	0·03	0·11	0·26	0·51	0·87	1·38	2·07	2·93	4·01
3400	0·00	0·03	0·10	0·23	0·45	0·77	1·23	1·83	2·60	3·56
3600	0·00	0·03	0·09	0·21	0·40	0·69	1·10	1·64	2·32	3·18
3800	0·00	0·02	0·08	0·18	0·36	0·62	0·98	1·47	2·09	2·86
4000	0·00	0·02	0·07	0·17	0·32	0·56	0·89	1·33	1·88	2·58

TABLE 21B.

FOR THE REDUCTION OF OBSERVATIONS OF THE OSCILLATIONS OF A "DAMPED" MAGNET NEEDLE (51 and 78.)

T and a = Time and amplitude of oscillation for ratio of damping k.
τ and a_1 = the corresponding time and amplitude without damping.
Then $T/\tau = \sqrt{1+\Lambda^2/\pi^2}$ and $a/a_1 = e^{\Lambda/\pi \tan^{-1} \pi/\Lambda} = k^{1/\pi \cdot \tan^{-1} \pi/\Lambda}$.

$\lambda = \log k$.	$\Lambda = \log \text{nat } k$.	$k = 10^\lambda = e^\Lambda$	$\sqrt{1+\Lambda^2/\pi^2}$.	$k^{1/\pi \tan^{-1} \pi/\Lambda}$.
0·00	0·0000	1·000	1·0000	1·0000
·01	·0230	1·023	1·0000	1·0115
·02	·0461	1·047	1·0001	1·0231
·03	·0691	1·072	1·0002	1·0347
·04	·0921	1·096	1·0004	1·0463
·05	·1151	1·122	1·0007	1·0578
·06	·1382	1·148	1·0010	1·0694
·07	·1612	1·175	1·0013	1·0811
·08	·1842	1·202	1·0017	1·0927
·09	·2072	1·230	1·0022	1·1044
·10	·2303	1·259	1·0027	1·1160
·11	·2533	1·288	1·0032	1·1277
·12	·2763	1·318	1·0039	1·1393
·13	·2993	1·349	1·0045	1·1510
·14	·3224	1·380	1·0052	1·1626
·15	·3454	1·413	1·0060	1·1743
·16	·3684	1·445	1·0069	1·1859
·17	·3914	1·479	1·0077	1·1975
·18	·4145	1·514	1·0087	1·2091
·19	·4375	1·549	1·0097	1·2208
·20	·4605	1·585	1·0107	1·2324
·21	·4835	1·622	1·0118	1·2440
·22	·5066	1·660	1·0130	1·2555
·23	·5296	1·698	1·0142	1·2670
·24	·5526	1·738	1·0155	1·2785
·25	·5756	1·778	1·0167	1·2900
·26	·5987	1·820	1·0180	1·3014
·27	·6217	1·862	1·0194	1·3128
·28	·6447	1·905	1·0208	1·3242
·29	·6677	1·950	1·0223	1·3356
·30	·6908	1·995	1·0239	1·3469
·31	·7138	2·042	1·0255	1·3582
·32	·7368	2·089	1·0271	1·3694
·33	·7599	2·138	1·0288	1·3806
·34	·7829	2·188	1·0306	1·3918
·35	·8059	2·239	1·0324	1·4029
·36	·8289	2·291	1·0342	1·4140
·37	·8520	2·344	1·0361	1·4250
·38	·8750	2·399	1·0381	1·4360
·39	·8980	2·455	1·0401	1·4469
·40	·9210	2·512	1·0421	1·4578
·41	·9441	2·570	1·0442	1·4686
·42	·9671	2·630	1·0463	1·4794
·43	0·9901	2·692	1·0485	1·4901
·44	1·0131	2·754	1·0507	1·5008

TERRESTRIAL MAGNETISM IN CENTRAL EUROPE FOR 1893.

(*From Neumayer's Tables and Maps.*)

TABLE 22.

HORIZONTAL INTENSITY IN C.M.G. UNITS.

The Horizontal Intensity increases yearly about 0·00015.

North Latitude.	Longitude East from Greenwich.										
	2°	4°	6°	8°	10°	12°	14°	16°	18°	20°	22°
45°	0·211	0·212	0·213	0·215	0·216	0·217	0·219	0·220	0·222	0·224	0·226
46	207	208	209	210	211	213	215	216	218	219	221
47	202	203	205	206	207	209	210	211	213	214	216
48	198	199	200	201	203	204	205	207	208	210	211
49	194	196	197	197	199	200	201	203	204	205	206
50	190	191	193	194	195	196	197	198	199	201	201
51	186	187	189	190	191	192	194	194	195	196	196
52	181	183	184	186	187	188	190	190	191	191	191
53	178	179	180	181	183	184	185	187	188	188	188
54	174	176	177	178	180	180	181	183	185	185	186
55	170	172	173	174	176	177	177	178	179	180	180

TABLE 22A.—HORIZONTAL INTENSITY IN THE UNITED STATES FOR 1885, IN C.M.G. UNITS.

North Latitude.	Longitude West from Greenwich.												
	65°	70°	75°	80°	85°	90°	95°	100°	105°	110°	115°	120°	125°
°	°	°	°	°	°	°	°	°	°	°	°	°	°
25													
26	...	...	...	·299									
27	...	...	...	·291									
28	...	...	...	·284	·295	·294	·298	·300	·301	·302	·303		
29	...	...	...	**·276**	·289	·287	·293	·294	·294	·295	·298		
30	...	...	...	·270	·282	·278	·286	·288	·288	·289	·293		
31	...	...	...	·265	·274	·272	·277	·281	·282	·283	·288		
32	...	...	...	·259	·266	·265	·269	·274	·276	·278	·283		
33	...	...	...	·254	·259	·259	·263	·269	·272	·273	·277	·280	
34	...	...	...	·246	**·252**	·252	·257	·263	·268	·269	·272	·274	
35	...	...	·230	·238	·244	·245	·250	·256	·262	·263	·266	·268	
36	...	...	·223	·229	·233	·237	·242	·249	·255	·257	·261	·263	
37	...	...	·216	·221	·223	·230	·236	·242	·248	·252	·255	·258	
38	...	...	·209	·213	·215	**·222**	·227	·234	·242	·246	·249	·252	
39	...	...	·195	·205	·208	·214	·220	·226	·233	·239	·243	·246	
40	...	...	·192	·197	·200	·205	**·211**	·219	·226	·233	·237	·240	·242
41	...	·182	·184	·189	·192	·196	·203	**·212**	·220	·226	·231	·235	·238
42	...	·174	·176	·180	·185	·187	·195	·204	**·213**	·219	·224	·230	·233
43	·163	·165	·168	·172	·174	·178	·187	·196	·206	·213	·218	·223	·228
44	·158	·158	·159	·161	·166	·170	·179	·189	·198	**·206**	·211	·216	·222
45	·152	·152	·153	·155	·158	·162	·170	·181	·190	·199	·205	·209	·214
46	·147	·146	·145	·149	·150	·155	·161	·171	·182	·192	·198	·202	·206
47	·140	·140	·133	·139	·140	·148	·154	·161	·175	·183	·190	·196	·200
48	...	...	...	·130	·133	·142	·147	·154	·166	·176	**·182**	·189	·194
49	...	...	...	·123	·124	·134	·139	·146	·158	·169	·176	·183	·188
50	...	...	...	...	·113	·125	·132	·139	...	...	·170	·177	·183

The thick figures represent a region of no annual change (in 1885). North of these the horizontal force is increasing; south of them, diminishing.

TABLE 23.—WEST DECLINATION.

North Latitude.	Longitude East from Greenwich.								
	5°	6°	7°	8°	9°	10°	11°	12°	13°
45°	13·5	13·1	12·7	12·4	12·0	11·6	11·1	10·7	10·3
50	14·3	13·8	13·3	12·9	12·4	11·9	11·4	11·0	10·4
55	14·7	14·4	14·0	13·4	12·8	12·3	11·7	11·1	10·4
	14°	15°	16°	17°	18°	19°	20°	21°	22°
45°	10·1	9·5	9·1	8·6	8·1	7·6	7·2	6·8	6·4
50	9·9	9·4	8·9	8·4	8·0	7·5	7·0	6·5	6·0
55	9·6	8·6	8·3	8·0	7·7	7·2	6·7	6·2	5·6

TABLE 23A.

DECLINATION IN THE UNITED STATES FOR 1890.

West declination is marked by a + sign ; east by a - sign.

North Latitude.	Longitude East from Greenwich.												
	65°	70°	75°	80°	85°	90°	95°	100°	105°	110°	115°	120°	125°
°	°	°	°	°	°	°	°	°	°	°	°	°	°
20	...	...	- 1·2	- 2·8	- 4·2	- 5·5	- 6·6	- 7·5	- 8·0				
25	...	...	+ 0·2	- 2·1	- 4·0	- 5·5	- 6·8	- 8·1	- 9·5	- 10·0	- 10·8		
30	...	...	...	- 1·1	- 4·0	- 5·7	- 7·4	- 9·3	- 10·6	- 11·9	- 12·0		
35	...	...	+ 3·8	- 0·2	- 2·4	- 5·7	- 8·4	- 11·1	- 12·6	- 13·7	- 14·5	- 15·0	
40	...	+ 10·5	+ 6·7	+ 3·0	- 2·3	- 5·5	- 9·2	- 12·1	- 14·1	- 15·7	- 17·0	- 17·6	- 17·6
45	+ 19·8	+ 17·0	+ 11·5	+ 5·2	+ 0·2	- 5·8	- 9·7	- 13·8	- 16·1	- 18·2	- 19·6	- 19·6	- 21·0
50	...	...	...	...	...	- 7·0	- 11·0	- 15·9	- 19·4	- 21·7	...	- 23·5	- 24·2

Tables 22A and 23A are obtained by graphic interpolation from the isodynamic and isoclinic charts published by the United States Coast and Geodetic Survey, in Appendix No. 6, 1885 ; and Table 24A similarly from Appendix No. 11, 1889, which have been kindly furnished by the U.S. Government. Declinations taken from Table 23A must be regarded as somewhat rough approximations, as in many parts of the States distribution of the isogonic curves is very irregular, and for exact information the original sources should be consulted.—H. R. P.

TABLE 24.—INCLINATION.

North Latitude.	Longitude East from Greenwich.				North Latitude.	Longitude East from Greenwich.			
	5°	10°	15°	20°		5°	10°	15°	20°
45°	62·4	61·6	61·0	60·5	51	66·6	66·1	65·7	65·2
46	3·1	2·3	1·8	1·4	52	7·2	6·7	6·4	6·1
47	3·8	3·1	2·6	2·3	53	7·9	7·3	7·0	6·8
48	4·6	3·9	3·3	3·0	54	8·5	8·0	7·7	7·5
49	5·3	4·7	4·1	3·8	55	9·2	8·7	8·4	8·1
50	6·0	5·5	5·0	4·5					

TABLE 24A.—INCLINATION IN THE UNITED STATES FOR 1885.

North Latitude.	Longitude West from Greenwich.												
	65°	70°	75°	80°	85°	90°	95°	100°	105°	110°	115°	120°	125°
°	°	°	°	°	°	°	°	°	°	°	°	°	°
25	...	...	...	55·2	54·5	53·6	52·6	51·8	51·3	50·4			
26	...	...	...	56·5	**55·6**	54·9	53·8	53·2	52·5	51·6			
27	...	...	...	57·7	56·9	56·1	55·0	54·4	53·7	52·9	51·6		
28	...	...	...	59·0	58·1	57·4	56·4	55·8	55·0	**53·9**	52·8		
29	...	...	...	60·4	59·3	58·5	57·7	57·1	**56·2**	55·2	54·0		
30	...	...	...	61·4	60·6	59·8	59·0	58·3	57·4	56·4	**55·1**		
31	...	...	...	62·5	61·7	**61·1**	**60·2**	**59·4**	58·5	57·5	56·5		
32	...	...	...	63·4	63·0	62·2	61·4	60·6	59·6	58·6	57·7		
33	...	...	...	64·4	63·8	63·4	62·4	61·6	60·7	59·7	58·8	57·8	
34	...	...	...	65·6	65·0	64·4	63·5	62·7	61·7	60·8	59·7	58·7	
35	...	...	66·8	66·7	66·0	65·4	64·5	63·7	62·8	61·8	60·7	**59·7**	
36	...	...	68·1	68·0	67·1	66·3	65·7	64·7	63·8	62·8	61·7	60·8	
37	...	...	69·0	68·7	68·1	67·5	66·8	65·8	64·9	63·9	62·8	61·9	
38	...	...	70·0	69·5	69·3	68·5	67·8	66·8	65·9	64·8	63·9	62·9	
39	...	...	70·9	70·5	70·0	69·4	68·8	67·8	66·9	65·8	64·8	64·0	**62·7**
40	...	...	71·7	71·5	71·2	70·3	69·7	69·0	68·0	66·8	65·7	64·7	63·5
41	...	...	72·8	72·4	72·0	71·3	70·8	70·0	68·9	67·8	66·6	65·6	64·3
42	...	73·2	73·5	73·0	72·8	72·4	71·9	71·1	69·9	68·8	67·6	66·5	65·1
43	...	73·9	74·1	73·9	73·6	73·2	72·7	71·9	70·9	69·7	68·6	67·3	66·1
44	...	74·6	74·8	75·0	74·5	74·1	73·6	72·7	71·8	70·7	69·5	68·4	67·1
45	74·8	75·4	75·6	75·7	75·4	75·0	74·4	73·5	72·6	71·5	70·4	69·4	68·0
46	75·8	76·1	76·4	76·4	76·3	75·9	75·2	74·4	73·4	72·3	71·2	70·3	68·9
47	76·7	76·6	77·3	77·1	77·0	76·7	75·9	75·2	74·2	73·1	71·9	70·9	69·8
48	77·5	77·2	...	77·9	78·0	77·4	76·7	75·9	75·0	73·8	72·7	71·6	70·8
49	...	...	...	78·8	78·8	78·2	77·5	76·7	75·7	74·7	73·4	72·3	71·7
50	...	...	...	79·7	79·7	79·0	78·3	77·5	76·5	75·4	74·3	73·1	72·6

The thick figures represent a region of no annual change (in 1885). North of these the dip is decreasing; south of them, increasing.

TABLE 25.—CONDUCTIVITY OF SOME METALS.

Most of the figures are only approximate.

Resistance increases at mean temperature for a rise of 1°
in mercury . . about 0·00090 of the whole
in pure solid metals . about 0·004 " "

In German silver, the temperature-coefficient varies from +0·00024 to 0·0006. This may be approximately estimated as 1-10000th of the conductivity as referred to mercury. Good "nickelin" 0·00023.

The resistance of alloys of 88% Cu with 12% Mn, as well as of 84Cu, 4Ni, and 12Mn ("Manganin"), and 60Cu with 40Ni ("Constantin"), is at mean temperatures almost independent of variations (Feussner and Lindeck). 25% nickel-copper ("patent nickel") has the temperature-coefficient +0·0002; 20% platinum-silver +0·00033. Compare note, p. 273.

The resistance of carbon diminishes with the temperature. The temperature-coefficient averages about 0·0002 to 0·0008.

The resistance of a wire of *lm* length, and *q* section is $=\sigma.l/q$ ohm, or $=s.l/q$ Siem, $s=1·06$. σ and $k=1/s$.

The numbers are for pure soft metal. Hardness, and especially impurities, lessen the conductivity.

At 18°.	Specific Resistance. A m.mm.²-wire has the resistance In ohms.	In Siem. units.	Conductivity as compared to Mercury at 0.
Silver	$\sigma=0·016$	$s=0·017$	$k=59$
Copper	0·0172	0·0182	55
Gold	0·023	0·024	41
Zinc	0·063	0·067	15
Iron	0·09 – 0·15	0·10 – 0·16	6 – 10
Steel	0·15 – 0·43	0·16 – 0·46	2 – 6
Platinum	0·14	0·15	6·5
Lead	0·21	0·22	4·6
Antimony	0·45	0·48	2·1
Mercury	0·958	1·016	0·984
Bismuth	1·2	1·2	0·8
Gas-carbon	50·0	50·0	0·02
Brass	0·07 – 0·09	0·07 – 0·10	10 – 14
20% Platinum-silver	0·20	0·21	4·8
German-silver	0·16 – 0·40	0·17 – 0·42	2·4 – 6
Nickelin	0·42	0·44	2·3
40% Nickel-copper	0·48	0·51	2·0
12% Manganese-copper	0·34	0·36	2·8
30% Manganese-copper	1·08	1·14	0·9

TABLE 26.

Electrical Conductivity of some Salts and Acids in Aqueous Solution at 18° referred to Mercury at 0°.

($ZnSO_4$ according to Beetz; KCl, NaCl, NH_4Cl, and HNO_3 according to Grotian and Kohlrausch; the others according to the author's observations.) Compare *Pogg. Ann.* clix. 257, and *Wied. Ann.* vi. 37.

By the percentage is meant the weight of the dissolved substance in 100 parts of the solution. The salts are anhydrous.

k is the conductivity at 18°, Δk the increase of k for 1° temperature, in percents of K_{18}.

Solution.	KCl.		NaCl.		NH_4Cl.		Na_2SO_4.		KSO_4.		$MgSO_4$.		$CuSO_4$.	
	$k.10^7$	Δk	$k.10^7$	Δk	$k.10^7$	Δk	$k.10^7$	Δk	$k.10^7$	Δk	$k.10^7$	Δk	$k.10^7$	Δk
5%	64	2·0	63	2·2	86	2·0	38	2·4	43	2·2	24	2·3	18	2·2
10	127	1·9	113	2·1	166	1·9	64	2·5	81	2·0	39	2·4	30	2·2
15	189	1·8	153	2·1	242	1·7	83	2·6	...	...	45	2·5	39	2·3
20	250	1·7	183	2·2	315	1·6	...	...	...	...	45	2·7	...	...
25	...	...	200	2·3	376	1·5	...	...	...	...	39	2·9	...	...

Solution.	HNO_3.		HCl.		H_2SO_4.		KI.		$ZnSO_4$.		$AgNO_3$.		KHO.	
	$k.10^7$	Δk	$k.10^7$	Δk	$k.10^7$	Δk	$k.10^7$	Δk	$k.10^7$	Δk	$k.10^7$	Δk	$k10^7$	Δk
5%	241	1·50	369	1·59	195	1·21	32	2·1	18	2·3	24	2·2	161	1·9
10	431	1·45	590	1·57	366	1·28	64	2·0	30	2·3	44	2·2	295	1·9
15	573	1·40	698	1·56	508	1·36	98	1·9	39	2·3	64	2·2	399	1·9
20	665	1·38	713	1·55	611	1·45	136	1·8	43	2·4	81	2·1	468	2·0
25	720	1·38	677	1·54	671	1·54	175	1·8	44	2·6	99	2·1	506	2·1
30	734	1·39	620	1·53	691	1·62	215	1·7	41	3·0	116	2·1	508	2·3
35	719	1·43	553	1·52	678	1·70	257	1·6	33	4·0	131	2·1	477	2·4
40	686	1·49	483	...	636	1·78	296	1·5	...	...	146	2·1	422	2·7
50	590	1·6	...	...	505	1·93	367	1·4	...	...	173	2·1	...	...
60	480	1·6	...	...	349	2·13	416	1·4	...	...	196	2·1	...	...
70	370	1·5	...	...	302	2·56	...	...	...	...	...	...	...	...
80	250	1·3	...	...	103	3·49	...	...	...	...	...	...	...	...

Maximum Conductivity of Solutions.

HNO_3	$k.10^7$=734	at 29·7 per cent.	Sp. gr.	1·185
HCl	717	at 18·3 ,,	,,	1·092
H_2SO_4	692	at 30·4 ,,	,,	1·224
KHO	510	at 28 ,,	,,	1·274
$MgSO_4$	46	at 17 ,,	,,	1·183
$ZnSO_4$	44·2	at 23·5 ,,	,,	1·286

TABLE 26A.—RESISTANCES OF METALS (J. C. Maxwell).

"In the following Table *R* is the resistance in ohms of a column 1 meter long, and 1 gram weight, at 0° C.; and *r* is the resistance in centimeters per second of a cube of 1 centimeter, according to the experiments of Matthiessen."

	Specific Gravity.	*R.*	*r.*	Percentage Increment of Resistance for 1° C. at 20° C.
Silver	10·50 hard drawn	0·1689	1609	0·377
Copper	8·95 ,,	0·1469	1642	0·388
Gold	19·27 ,,	0·4150	2154	0·365
Lead	11·391 pressed	2·257	19847	0·387
Mercury	13·595 liquid	13·071	96146	0·072
Gold 2, Silver 1	15·218 hard or annealed	1·668	10988	0·065
Selenium at 100° C.	crystalline form	...	6×10^{13}	1·00

TABLE 26B.—ELECTROMOTIVE FORCE OF CONSTANT BATTERIES (J. C. Maxwell).

			Concentrated Solution of	Volt.
Daniell I.	Amalgamated Zinc	H_2SO_4 + 4 Aq	$CuSO_4$ Copper	1·079
,, II.	,,	,, +12 Aq	$CuSO_4$,,	0·978
,, III.	,,	,, + ,,	$CuNO_3$,,	1·00
Bunsen I.	,,	,, + ,,	HNO_3 Carbon	1·964
,, II.	,,	,, + ,,	sp. gr. 1·38 ,,	1·888
Grove	,,	,, + 4 Aq	HNO_3 Platinum	1·956

TABLE 27.—ELECTRO-CHEMICAL EQUIVALENTS.

A current of 1 ampere = 0·1 [cm. g. sec.] = 10 [mm. mg. sec.] decomposes or separates

	Mg. silver.	Mg. copper.	Mg. hydrogen.	Mg. water.	C.c. mixed gases at 0° C. and 760 mm.
In 1 sec.	1·1181	0·3284	0·01039	0·0933	0·1740
In 1 min.	67·09	19·70	0·623	5·60	10·44
In 1 hr.	4025	1182	37·41	336	626

TABLE 27A.—COMPARISON OF MEASURES OF ELECTRIC CURRENT-STRENGTH.

A Current-Strength which is measured in	Must be multiplied by the following Numbers to reduce it to				
	Cubic Cm. Water Gases per Minute.	Mgr. Water per minute.	Mgr. Copper per Minute.	Mgr. Silver per Minute.	Magnetic Measure $\frac{\text{Mm.}^{\frac{1}{2}}\ \text{Mgr.}^{\frac{1}{2}}}{\text{Sec.}}$
Cub. Cm. Water Gases per min.	...	0·5363	1·889	6·432	0·9484
Mgr. Water per Min.	1·865	...	3·522	11·99	1·769
Mgr. Copper ,,	0·5294	0·2839	...	3·405	0·5023
Mgr. Silver ,,	0·1555	0·0834	0·2937	...	0·1475
Magnetic Measure $\frac{\text{Mm.}^{\frac{1}{2}}\ \text{Mgr.}^{\frac{1}{2}}}{\text{Sec.}}$	1·054	0·5653	1·991	6·779	...

TABLE 27B.—MOLECULAR CONDUCTIVITY k/μ OF SOME ELECTROLYTES AT 18° C. in aqueous solution containing μ equivalents. [F. K., *Wied. Ann.* xxvi. 195, 1885.]

μ is the strength per liter in gram-equivalents ; compare Table 3A.
k is the conductivity as compared to mercury at 0° C.
The limiting values are graphically extrapolated, as well as the bracketed values for acids, upon the hypothesis that the first observed conductivities are depressed by the impurity of the water.

	KCl.	NaCl.	HCl.	$\frac{1}{2}K_2SO_4$.	$\frac{1}{2}MgSO_4$.	$\frac{1}{2}H_2SO_4$.
	$10^8\frac{k}{\mu}$	$10^8\frac{k}{\mu}$	$10^8\frac{k}{\mu}$	$10^8\frac{k}{\mu}$	$10^8\frac{k}{\mu}$	$10^8\frac{k}{\mu}$
Approximate Limiting Value . . .	1230	1040	[3550]	1290	[1100]	[3800]
μ=0·0001	1215	1026	[3500]	1125	1034	[3580]
0·0002	1210	1021	[3490]	1240	1015	[3520]
0·0005	1201	1016	[3480]	1224	976	[3440]
0·001	1193	1008	3460	1207	935	[3350]
0·002	1185	999	3455	1181	881	3250
0·005	1165	981	3445	1140	790	3050
0·01	1147	962	3416	1098	715	2860
0·02	1123	938	3390	1044	632	2650
0·05	1083	897	3330	959	534	2340
0·1	1047	865	3240	897	467	2090
0·2	1009	826	3140	832	408	1960
0·5	958	757	3020	736	330	1900
1	917	696	2780	672	271	1820
2	864	604	2340	...	202	1700
5	...	398	1420	...	82	1270
10	...	...	600	...	...	655
20	...	...	...	...	...	147
30	...	...	...	...	...	30

TABLE 27C.—ELECTRICAL POTENTIAL AND STRIKING DISTANCE S IN AIR.

(From Baille, Quincke, Bichat and Blondlot, Paschen, Freyberg.)

R = Radius of ball in cm. V = Electrostatic potential in [cm. g. sec.]
300 V. gives the potential in volts.

S	R in cm.			S	R in cm.		
	1·0	0·5	3·0		1·0	0·5	3·0
cm.	V.	V.	V.	cm.	V.	V.	V.
0·02	5·2	5·1	...	0·4	48	48	45
0·04	8·2	8·1	...	0·5	57	57	55
0·06	11·0	10·8	...	0·6	65	66	65
0·08	13·5	13·3	...	0·7	72	75	75
0·1	16	16	15	0·8	78	83	87
0·2	28	28	26	0·9	82	91	98
0·3	38	38	36	1·0	85	97	109

TABLE 28.—Units of the Absolute System of Measurement.

Fundamental magnitudes :—Length l, mass m, time t.

Each kind of magnitude is expressed in the form l^λ, m^μ, t^τ; λ, μ, τ, being the dimensions of the magnitude in respect to length, mass, and time.

The unit of time is the second.

Related units of length and mass are : *dm.k.; cm.g. ; mm.mg.*

The numbers in the last column show the ratio N in which a unit increases by the change from *mm.mg.* to *cm.g.;* or from *cm.g.* to *dm.k.*

Quantities given in *mm.mg.* units (Gauss-Weber system) must therefore be divided by N to reduce them to *cm.g.*

The middle column contains the names of various units in use, and the number of them which make up the corresponding *cm. g.* unit.

If we put 300 × 10 *cm.* sec., then the ratio Electromag. unit/Electrostat. unit for electric quantity or current is v; for electric potential $1/v$; for electric capacity v^2; and for electric resistance $1/v^2$.

	λ	μ	τ		N
Angle	0	0	0		1
Length	1	0	0	1 centimeter	10
Linear curvature	1	0	0		10^{-1}
Area of surface	2	0	0		10^2
Volume	3	0	0		10^3
Mass	0	1	0	1 gram	10^3
Density	−3	1	0		1
Time, period of oscillation	0	0	1	1 second	1
Velocity	1	0	−1		10
Angular velocity	0	0	−1		1
Acceleration	1	0	−2		10
Angular acceleration	0	0	−2		1
Force	1	1	−2	1 dyne	10^4
Moment of rotation, directive force	2	1	−2		10^5
Pressure	−1	1	−2		10^2
Modulus elasticity	−1	1	−2		10^2
Capillary constant	0	1	−2		10^3
Coefficient internal friction	−1	1	−1		10^2
Moment of inertia	2	1	0		10^5
Work, energy, *vis viva* quantity of heat	2	1	−2	1 erg.	10^5
Number of oscillations, pitch	0	0	−1		1
Index of refraction	0	0	0		1
Rotatory power	−1	0	0		10^{-1}
Electrostatic System.					
Electric quantity	$\frac{3}{2}$	$\frac{1}{2}$	−1	$\frac{1}{3}10^{-9}$ coulombs	10^3
Potential	$\frac{1}{2}$	$\frac{1}{2}$	−1	300 volts	10^2
Capacity	1	0	0	$\frac{1}{9}$ 10^{-11} farads	10
Dielectric constant	0	0	0		1
Electric current	$\frac{3}{2}$	$\frac{1}{2}$	−2	$\frac{1}{3}$ 10^{-9} am.	10^3
Resistance	−1	0	1	9×10^{11} ohm.	10^{-1}

TABLE 28—*continued.*

	λ	μ	τ		N
ELECTRO-MAGNETIC SYSTEM					
Magnetic pole	$\frac{3}{2}$	$\frac{1}{2}$	-1		10^3
Magnetic potential . .	$\frac{1}{2}$	$\frac{1}{2}$	1		10^2
Magnetism of bar . . .	$\frac{5}{2}$	$\frac{1}{2}$	-1		10^4
Magnetic intensity . .	$\frac{1}{2}$	$\frac{1}{2}$	-1		10
Spec. Magnetism (vol.) . .	$-\frac{1}{2}$	$\frac{1}{2}$	-1		10
Const. of magnetisation . .	0	0	0		1
Electric current . . .	$\frac{1}{2}$	$\frac{1}{2}$	-1	10 amperes	10^2
Current density . . .	$-\frac{3}{2}$	$\frac{1}{2}$	-1		1
Electric quantity . . .	$\frac{1}{2}$	$\frac{1}{2}$	0	10 coulombs	10^2
Electro-chemical equivalent .	$-\frac{1}{2}$	$\frac{1}{2}$	0		10
Electro-magnetic force, potential	$\frac{3}{2}$	$\frac{1}{2}$	-2	10^{-8} volts	10^8
Capacity	-1	0	2	10^9 farads	10^{-1}
Resistance	1	0	1	10^{-9} ohm	10
Specific resistance . . .	2	0	-1		10^3
Current efficiency . . .	2	1	-3	10^{-7} watts	10^5
El. dyn. potential / Self-induction coefficient } .	1	0	0		10

TABLE 28B.—BIRMINGHAM WIRE GAUGE (Holtzapffel).

BWG.	Diameter in inches.	BWG.	Diameter in inches.	BWG.	Diameter in inches.
0	0·340	14	0·083	28	0·014
2	·284	16	·065	30	·012
4	·238	18	·049	32	·009
6	·203	20	·035	33	·008
8	·165	22	·028	34	·007
10	·134	24	·022	35	·005
12	·109	26	·018	36	·004

TABLE 28C.—MEAN SPECIFIC HEATS OF WATER AND PLATINUM.

Water (Regnault).		Platinum (Pouillet).	
From 0° to 40° C.	1·0013	From 0° to 100° C.	0·0335
" 0 " 80	1·0035	" 0 " 300	0·0343
" 0 " 120	1·0067	" 0 " 500	0·0352
" 0 " 160	1·0109	" 0 " 700	0·0360
" 0 " 200	1·0160	" 0 " 1000	0·0373
" 0 " 230	1·0204	" 0 " 1200	0·0382

TABLE 29.—Symbols, Atomic Weight, Valence, and Specific Heat of some Elements.

The Atomic Weight is the smallest proportion in which the element enters into combination; hydrogen being taken as 1, or practically oxygen as 16.

The Valence or Atomicity indicates the largest number of atoms of hydrogen or other univalent element which one atom usually replaces or combines with. In many cases the element acts as if its valence were less by 2 or 4 than that given. Equal quantities of electricity liberate equal valences.

Specific Heat multiplied by atomic weight is nearly constant for the same physical state in all elements.

The Electro-negative elements, or those which in electrolysis appear at the positive pole or zincode, are printed in italics; the electro-positive, or those which appear at the negative pole or platinode, in Roman type. The difference, however, is only one of degree.

Name.	Symbol.	Atomic Weight.	Valence.	Specific Heat of Equal Parts.
Aluminium	Al	27·1	VI. III.	0·202
Antimony	Sb	120·3	V.	0·0508
Arsenic	As	75·0	V.	0·0814
Barium	Ba	137·0	II.	......
Bismuth	Bi	208·0	V.	0·0308
Boron	B	11·0	IV.	0·25
Bromine	Br	79·96	I.	0·1060 Liquid 0·0843 Solid
Cadmium	Cd	112·1	II.	0·0548
Calcium	Ca	40·0	II.	......
Carbon	C	12·0	IV.	0·459 Charcoal 0·467 Diamond
Chlorine	Cl	35·45	I.	0·1210 Const. press.
Chromium	Cr	52·2	VI.	0·100
Cobalt	Co	59·0 ?	VI.	0·1070
Copper	Cu	63·3	II.	0·0952
Fluorine	F	19·0	I.	......
Gold	Au	197·2	III.	0·0324
Hydrogen	H	1·003	I.	0·4090 Const. press.
Iodine	I	126·86	I.	0·0541
Iron	Fe	56·0	VI. III.	0·1138
Lead	Pb	206·9	IV.	0·0314
Lithium	Li	7·03	I.	0·941
Magnesium	Mg	24·4	II.	0·2499
Manganese	Mn	55·1	VI. III.	0·122
Mercury	Hg	200·4	II.	0·0319 Solid
Nickel	Ni	58·5	VI. III.	0·108
Nitrogen	N	14·04	V.	0·2438 Const. press.
Oxygen	O	16·00	II.	0·2175 Const. press.
Phosphorus	P	31·0	V.	0·170–0·189
Platinum	Pt	194·8	IV.	0·0324
Potassium	K	39·14	I.	0·166
Selenium	Se	79·1	VI.	0·074–0·086
Silicon	Si	28·4	IV.	0·203
Silver	Ag	107·94	I.	0·0560
Sodium	Na	23·06	I.	0·2934
Strontium	Sr	87·5	II.	......
Sulphur	S	32·06	VI.	0·163–0·178
Tin	Sn	118·1	IV.	0·0548–0·0562
Zinc	Zn	65·4	II.	0·0932–0·0955

TABLE 30.

Geographical Position and Height of some Places above the Sea.

East longitude reckoned from Berlin is about 13·39 smaller, and from Ferro 17·66 larger than Greenwich.

	East from Greenwich.	North Latitude.	Height above Sea.		East from Greenwich.	North Latitude.	Height above Sea.
Aix-la-Chapelle	6·1	50·78	160–200	Heidelberg .	8·7	49·41	100
Amsterdam .	4·9	52·37	...	Jena . . .	11·6	50·94	160
Basle . . .	7·6	47·56	260	Innsbruck .	11·4	47·27	570
Berlin . . .	13·4	52·50	40	Kiel	10·2	54·34	...
Berne . . .	7·4	46·95	550	Königsberg .	20·5	54·71	...
Birmingham .	−1·9	52·29	140–145	Leeds . . .	−1·5	53·78	60–100
Bonn . . .	7·1	50·73	50	Leipzig . .	12·4	51·34	100
Boston . .	−70·0	42·37	...	Madrid . .	−3·7	40·41	660
Bremen . . .	8·8	53·08	...	Marburg . .	8·8	50·81	180–240
Breslau . .	17·0	51·11	130	Milan . . .	9·2	45·47	130
Brunswick .	10·5	52·27	100	Montreal . .	−73·6	45·50	...
Brussels . .	4·4	50·85	90	Munich . .	11·6	48·15	530
Cambridge .	0·1	52·20	...	Münster . .	7·6	51·97	60
Cologne . .	7·0	50·94	40	New York .	−74·0	40·71	...
Copenhagen .	12·6	55·69	...	Paris . . .	2·3	48·83	60
Carlsruhe . .	8·4	49·01	120	Pesth . . .	19·1	47·50	70
Cassel . . .	9·5	51·32	160	Philadelphia .	−75·0	40·00	...
Dantzig . .	18·7	54·35	...	Prague . . .	14·4	50·09	200
Darmstadt .	8·7	49·87	140	Rome . . .	12·4	41·90	30
Dorpat . .	26·7	58·38	50	San Francisco	−122·4	37·77	...
Dresden . .	13·7	51·04	100	Stockholm .	18·1	59·34	...
Edinburgh .	−3·2	55·95	...	St. Petersburg	30·3	59·94	...
Freiburg in B.	7·8	47·96	280	Strassburg .	7·8	48·58	150
Giessen . .	8·6	50·59	140	Stuttgart . .	9·2	48·78	270
Glasgow . .	−4·3	55·85	...	Sydney . .	151·1	33·85	...
Göttingen . .	9·9	51·53	130	Tübingen . .	9·1	48·52	320–380
Greenwich .	0·0	51·48	...	Vienna . .	16·4	48·23	180
Halle . . .	12·0	51·49	100	Washington .	−77·0	38·89	...
Hamburg . .	10·0	53·55	...	Würzburg . .	9·9	49·79	170
Hanover . .	9·7	52·38	70	Zurich . . .	8·6	47·38	420–500

TABLE 31.

Declination of the Sun, Equation of Time, and Sidereal Time for Mean Noon of 15° E. from Greenwich.

(Central European "Unit Time.")

Sidereal Time at Noon increases daily 3 min. 56·6 sec. = 236·6 sec.
Mean time of a place = Solar Time + Equation of Time.

The figures in brackets are for Leap Year.

Day.	Declination of the Sun.	Difference for 1 day.	Equation of Time.	Sidereal Time at Noon.	Day	Declination of the Sun	Difference for 1 day.	Equation of Time.	Sidereal Time at Noon.
	°	°	m. s.	h. m. s.		°	°	m. s.	h. m. s.
Jan.					July				
0 (1)	- 23·10		+ 3 25	18 38 42	4	+22·92	0·068	+ 4 0	6 48 4
5 (6)	– 22·64	0·092	+ 5 34	18 58 24	9	+22·41	0·102	+ 4 49	7 7 47
10 (11)	- 21·99	0·130	+ 7 42	19 18 7	14	+21·73	0·136	+ 5 29	7 27 30
15 (16)	21·16	0·166	+ 9 36	19 37 50	19	+20·91	0·164	+ 5 58	7 47 13
20 (21)	- 20·16	0·200	+11 13	19 57 33	24	+19·94	0·194	+ 6 13	8 6 56
25 (26)	– 19·01	0·230	+12 33	20 17 16	29	+18·88	0·222	+ 6 13	8 26 38
30 (31)	- 17·71	0·260	+13 32	20 36 58	Aug.				
Feb.					3	+17·59	0·248	+ 5 57	8 46 21
4 (5)	- 16·27	0·288	+14 10	20 56 41	8	+16·23	0·272	+ 5 27	9 6 4
9 (10)	– 14·73	0·308	+14 27	21 16 24	13	+14·76	0·294	+ 4 42	9 25 47
14 (15)	– 13·08	0·330	+14 25	21 36 7	18	+13·19	0·314	+ 3 44	9 45 29
19 (20)	11 34	0·348	+14 5	21 55 49	23	+11·54	0·330	+ 2 33	10 5 12
24 (25)	– 9·52	0·364	+13 28	22 15 32	28	+ 9·81	0·346	+ 1 11	10 24 55
March					Sept.				
1	- 7·65	0·374	+12 36	22 35 15	2	+ 8·01	0·360	– 0 20	10 44 38
6	– 5·73	0·384	+11 31	22 54 58	7	+ 6·16	0·370	1 59	11 4 21
11	– 3 78	0·390	+10 15	23 14 41	12	+ 4·27	0·378	– 3 41	11 24 3
16	1·81	0·394	+ 8 52	23 34 23	17	+ 2·35	0·384	– 5 26	11 43 46
21	+ 0·16	0·394	+ 7 23	23 54 6	22	+ 0·40	0·390	– 7 12	12 3 29
26	+ 2·13	0·394	+ 5 52	0 13 49	27	– 1·55	0·390	– 8 55	12 23 12
31	+ 4 08	0·390	+ 4 19	0 33 32	Oct.				
April					2	– 3·49	0·388	–10 34	12 42 54
5	+ 6·00	0·384	+ 2 49	0 53 14	7	5·42	0·386	–12 4	13 2 37
10	+ 7·87	0·374	+ 1 23	1 12 57	12	– 7·32	0·380	–13 24	13 22 20
15	+ 9·69	0·364	+ 0 4	1 32 40	17	– 9·19	0·374	–14 31	13 42 3
20	+11·45	0·352	– 1 5	1 52 23	22	–10·99	0·360	–15 23	14 1 45
25	+13·12	0·334	– 2 4	2 12 5	27	–12·73	0·348	–16 0	14 21 28
30	+14·71	0·318	– 2 52	2 31 48	Nov.				
May					1	14·33	0·330	–16 18	14 41 11
5	+16·19	0·296	– 3 27	2 51 31	6	–15·94	0·312	–16 16	15 0 54
10	+17·57	0·276	– 3 48	3 11 14	11	–17·33	0·288	–15 52	15 20 37
15	+18·82	0·250	– 3 53	3 30 57	16	–18·71	0·266	–15 7	15 40 19
20	+19·94	0·224	– 3 45	3 50 39	21	19·89	0·236	–14 2	16 0 2
25	+20·92	0·196	3 23	4 10 22	26	–20·92	0·206	–12 36	16 19 45
30	+21·74	0·164	– 2 49	4 30 5	Dec.				
June					1	21·79	0·174	–10 53	16 39 28
4	+22·12	0·136	– 2 4	4 49 48	6	–22·49	0·140	– 8 54	16 59 10
9	+22 92	0·100	– 1 11	5 9 30	11	–23·00	0·102	– 6 40	17 18 53
14	+23·26	0·068	– 0 10	5 29 13	16	–23·32	0·064	4 17	17 38 36
19	+23·43	0·034	+ 0 55	5 48 56	21	–23·45	0·026	– 1 49	17 58 19
24	+23·43	0·002	+ 2 0	6 8 39	26	–23·39	0·012	+ 0 41	18 18 2
29	+23·26	0·034	+ 3 2	6 28 22	31	–23·12	0·054	+ 3 8	18 37 44

TABLE 32.

Correction Table for the Beginning of the Year.

Year.	Correction.	Year.	Correction.
1891 . . .	+0·151	1897 . . .	+0·699
1892 . . .	+0·909	1898 . . .	+0·457
1893 . . .	+0·667	1899 . . .	+0·214
1894 . . .	+0·425	1900 . . .	−0·028
1895 . . .	+0·182	1901 . . .	−0·270
1896 . . .	+0·941	1902 . . .	−0·512

TABLE 33.

Sun's Radius.

Date.	Radius.	Date.	Radius.
January 1 . .	0·272	July 1 . . .	0·263
February 1 . .	0·271	August 1 . . .	0·263
March 1 . . .	0·269	September 1 . .	0·265
April 1 . . .	0·267	October 1 . .	0·267
May 1 . . .	0·265	November 1 . .	0·269
June 1 . . .	0·263	December 1 . .	0·271

TABLE 34.

Refraction at Different Altitudes.

Altitude.	Refraction.	Altitude.	Refraction.
°	°	°	°
5 . . .	0·16	50 . . .	0·013
10 . . .	0·09	60 . . .	0·009
15 . . .	0·06	70 . . .	0·006
20 . . .	0·04	80 . . .	0·003
30 . . .	0·028	90 . . .	0·000
40 . . .	0·019		

TABLE 35.

Mean Places of some Principal Stars for 1893·0.

	Right Ascension.			Yearly Variation.	Declination.			Yearly Variation.
	hr.	min.	sec.	sec.	°	′	″	″
α Cassiopeiæ	0	34	26·1	+3·37	55	57	1	+19·8
α Arietis	2	1	8·4	+3·37	22	57	23	+17·2
α Persei	3	16	41·0	+4·26	49	28	48	+13·1
α Tauri (Aldebaran)	4	29	46·8	+3·44	16	17	37	+ 7·5
α Aurigæ (Capella)	5	8	47·1	+4·43	45	53	19	+ 4·0
α Orionis	5	49	22·7	+3·25	7	23	12	+ 1·0
α Canis Majoris (Sirius)	6	40	26·1	+2·64	−16	34	11	− 4·7
α Geminorum (Castor)	7	27	46·2	+3·84	32	7	22	− 7·6
α Canis Minoris (Procyon)	7	33	42·1	+3·14	5	29	56	− 9·0
α Hydræ	9	22	19·8	+2·95	−8	11	42	−15·4
α Leonis (Regulus)	10	2	40·4	+3·20	12	29	24	−17·5
α Ursæ Majoris	10	57	7·4	+3·75	62	19	43	−19·4
β Leonis	11	43	36·1	+3·06	15	10	13	−20·1
α Virginis (Spica)	13	19	33·3	+3·15	−10	36	10	−18·9
α Bootis (Arcturus)	14	10	46·8	+2·73	19	44	23	−18·9
α Coronæ (Gemma)	15	30	9·5	+2·54	27	4	30	−12·3
α Scorpii (Antares)	16	22	50·8	+3·67	−26	11	40	− 8·3
α Ophiuchi	17	29	58·0	+2·78	12	38	17	− 2·8
α Lyræ (Vega)	18	33	18·9	+2·03	38	41	3	+ 3·2
α Aquilæ (Altair)	19	45	33·7	+2·93	8	35	9	+ 9·3
α Cygni	20	37	47·1	+2·04	44	53	53	+12·7
α Piscis Australis (Fomalhaut)	22	51	44·2	+3·32	−30	11	22	+19·0
α Pegasi	22	59	25·8	+2·98	14	37	47	+19·3
α Ursæ Minoris (Polaris)	1	19	41·7	+23·9	88	44	15	+18·9
δ Ursæ Minoris	18	6	49·2	−19·5	86	36	44	+ 0·6

TABLE 36.

NUMBERS FREQUENTLY REQUIRED.

(The fractions in brackets are approximate values.)

$\pi = 3{\cdot}1416\ (\frac{22}{7})$, $\pi^2 = 9{\cdot}8696$, $\frac{1}{\pi} = 0{\cdot}31831$, $\frac{\pi}{4} = {\cdot}7854$, log. $\pi = 0{\cdot}4971499$.

Basis of natural logarithms $e = 2{\cdot}7183$; log. $e = 0{\cdot}43429$.
The modulus of natural logarithms M = 1/log. $e = 2{\cdot}3026$; log. M = ·36222.
The angle of which the arc is equal to the radius $= 57^\circ{\cdot}2958 = 3437{\cdot}75' = 206265''$.
Ratio of the probable to the mean error = 0·6745 ($\frac{2}{3}$).

1 Paris foot	= 0·32484 meter ($\frac{13}{40}$).	1 meter	= 3·0784 Paris feet.
1 Paris line	= 2·2588 mm. ($\frac{9}{4}$).	1 mm.	= 0·44330 Paris line.
1 Rhenish foot	= 0·31385 meter ($\frac{10}{32}$).	1 meter	= 3·1862 Rhenish feet.
1 English foot	= 0·30479 ,, ($\frac{7}{23}$).	1 meter	= 3·2809 English feet.
1 Geogr. mile	= 7·4204 kilom. ($\frac{30}{4}$).	1 kilom.	= 0·13476 Geogr. mile.
1 English mile	= 1·60929 ,,	1 kilom.	= 0·62138 English mile ($\frac{5}{8}$).

Half the major axis of the earth = 6378·2 kilometers.
,, minor ,, = 6356·5 ,,
The mean semidiameter of earth = 6367·4 ,,

Mean length of civil year, 365 days 5 hours 48·8 min.

Sidereal day = mean day − 3 min. 55·9 sec. = 0·99727 mean day.

Velocity of sound at 0° C. in dry air $= 331\ \frac{\text{meter}}{\text{sec.}}$.

Coefficient of expansion of gases 0·00367 ($\frac{1}{273}$).

1 grm. wt. at 45° lat. = 980·6 cm.g. sec.$^{-2}$.

1 at. pressure = 1033 g.-wt./cm.^{2} = 1013200 cm.$^{-1}$ g. sec.$^{-2}$.

1 water gram. calorie = 428 g. wt. m = 42000000 cm.^{2}g. sec.$^{-2}$.

Latent heat of water = 79·9; of steam = 536 (1 lb. water 1° Fah. = 772 foot-pounds.)

Specific heat of air at constant pressure = 0·237.

Ratio of specific heats of air or H ; constant pressure to constant vol. = 1·40.

Capillary constant of water, 7·8 ; alcohol, 2·3 ; mercury, 50 mg./mm.

Ratio of molecular wt. to vapour density (air) = 28·9 ; to H = 1.

1 liter H. at 0° C. and 760 mm. weighs 0·0896 g.

Velocity of light *in vacuo* = 300000 km./sec.

Index of refraction of air = 1·00029.

Wave-length of Na light (D. Fraunhofer) = 0·0005893 mm.

A quartz plate 1 mm. thick rotates Na light 21°·7.

Ratio of electro-magnetic to electrostatic unit of electricity $v = 300{\cdot}10^8$ cm./sec.

1 Ampere = 0·1 cm.$^{\frac{1}{2}}$ g.$^{\frac{1}{2}}$ sec.$^{-1}$ el.-mag. = 300×10^7 cm.$^{\frac{3}{2}}$ g.$^{\frac{1}{2}}$ sec.$^{-2}$ el.-stat. units/sec. = 1·1181 mg. Ag/sec.

1 Ohm = 10^9 cm. sec.$^{-1}$ el.-mag. = $1/9 \times 10^{-11}$ cm.$^{-1}$ sec. el.-stat. = 1·063 m./mm.^{2} Hg. 0° C. = 1·014 B. A. units.

1 Volt = 10^8 cm.$^{\frac{3}{2}}$ g.$^{\frac{1}{2}}$ sec.$^{-2}$ el.-mag. = 1/300 cm.$^{\frac{1}{2}}$ g.$^{\frac{1}{2}}$ sec.$^{-1}$ el.-stat. (Ger. legal Ohm and Volt see, p. 423.)

Bunsen cell = 1·9 ; Daniell = 1·1 to 1·2 ; Clark at 15° = 1·433 volts.

1 Volt-ampere or Watt = 0·102, kg.-wt. m./sec. = 0·24 water g. cal./sec.

TABLE 37.—Squares, Square Roots, and Reciprocals.

Conversion of degrees of Arc into absolute Angular Measure.
Table for use with the Wheatstone-Kirchhoff Bridge.

n.	n^2.	$\sqrt{n}$.	$\frac{1}{n}$	$n\frac{\pi}{180}$	$\frac{n}{100-n}$
1	1	1·000	1·0000	0·0175	0·0101
2	4	1·414	0·5000	0·0349	0·0204
3	9	1·732	0·3333	0·0524	0·0309
4	16	2·000	0·2500	0·0698	0·0417
5	25	2·236	0·2000	0·0873	0·0526
6	36	2·449	0·1667	0·1047	0·0638
7	49	2·646	0·1429	0·1222	0·0753
8	64	2·828	0·1250	0·1396	0·0870
9	81	3·000	0·1111	0·1571	0·0989
10	100	3·162	0·1000	0·1745	0·1111
11	121	3·317	0·0909	0·1920	0·1236
12	144	3·464	0·0833	0·2094	0·1364
13	169	3·606	0·0769	0·2269	0·1494
14	196	3·742	0·0714	0·2443	0·1628
15	225	3·873	0·0667	0·2618	0·1765
16	256	4·000	0·0625	0·2793	0·1905
17	289	4·123	0·0588	0·2967	0·2048
18	324	4·243	0·0556	0·3142	0·2195
19	361	4·359	0·0526	0·3316	0·2346
20	400	4·472	0·0500	0·3491	0·2500
21	441	4·583	0·0476	0·3665	0·2658
22	484	4·690	0·0455	0·3840	0·2821
23	529	4·796	0·0435	0·4014	0·2987
24	576	4·899	0·0417	0·4189	0·3158
25	625	5·000	0·0400	0·4363	0·3333
26	676	5·099	0·0385	0·4538	0·3514
27	729	5·196	0·0370	0·4712	0·3699
28	784	5·292	0·0357	0·4887	0·3889
29	841	5·385	0·0345	0·5061	0·4085
30	900	5·477	0·0333	0·5236	0·4286
31	961	5·568	0·0323	0·5411	0·4493
32	1024	5·657	0·0313	0·5585	0·4706
33	1089	5·745	0·0303	0·5760	0·4925
34	1156	5·831	0·0294	0·5934	0·5152
35	1225	5·916	0·0286	0·6109	0·538
36	1296	6·000	0·0278	0·6283	0·562
37	1369	6·083	0·0270	0·6458	0·587
38	1444	6·164	0·0263	0·6632	0·613
39	1521	6·245	0·0256	0·6807	0·639
40	1600	6·325	0·0250	0·6981	0·667
41	1681	6·403	0·0244	0·7156	0·695
42	1764	6·481	0·0238	0·7330	0·724
43	1849	6·557	0·0233	0·7505	0·754
44	1936	6·633	0·0227	0·7679	0·786
45	2025	6·708	0·0222	0·7854	0·818
46	2116	6·782	0·0217	0·8029	0·852
47	2209	6·856	0·0213	0·8203	0·887
48	2304	6·928	0·0208	0·8378	0·923
49	2401	7·000	0·0204	0·8552	0·961
50	2500	7·071	0·0200	0·8727	1·000

TABLE 37—*continued.*—Squares, Square Roots, & Reciprocals.

Conversion of degrees of Arc into absolute Angular Measure.
Table for use with the Wheatstone-Kirchhoff Bridge.

n.	n^2.	$\sqrt{n}$.	$\frac{1}{n}$.	$n\frac{\pi}{180}$.	$\frac{n}{100-n}$.
50	2500	7·071	0·0200	0·873	1·000
51	2601	7·141	0 0196	0·890	1·041
52	2704	7·211	0·0192	0·908	1·083
53	2809	7·280	0·0189	0·925	1·128
54	2916	7·348	0·0185	0·942	1·174
55	3025	7·416	0·0182	0·960	1·222
56	3136	7·483	0·0179	0·977	1·273
57	3249	7·550	0·0175	0·995	1·326
58	3364	7·616	0·0172	1·012	1·381
59	3481	7·681	0·0169	1·030	1·439
60	3600	7 746	0·0167	1·047	1·500
61	3721	7·810	0·0164	1·065	1·564
62	3844	7·874	0·0161	1·082	1·632
63	3969	7·937	0·0159	1 100	1·703
64	4096	8·000	0·0156	1·117	1·778
65	4225	8·062	0·0154	1·134	1·857
66	4356	8·124	0·0152	1·152	1·941
67	4489	8·185	0·0149	1 169	2·030
68	4624	8·246	0·0147	1·187	2·125
69	4761	8·307	0·0145	1·204	2·226
70	4900	8·367	0·0143	1·222	2·333
71	5041	8·426	0·0141	1·239	2·448
72	5184	8·485	0·0139	1·257	2·571
73	5329	8·544	0·0137	1·274	2·704
74	5476	8·602	0·0135	1·292	2·846
75	5625	8·660	0·0133	1·309	3·000
76	5776	8·718	0·0132	1·326	3·167
77	5929	8·775	0·0130	1 344	3·348
78	6084	8·832	0·0128	1·361	3·545
79	6241	8·888	0 0127	1·379	3·762
80	6400	8·944	0 0125	1·396	4·00
81	6561	9·000	0·0123	1·414	4·26
82	6724	9·055	0·0122	1·431	4·56
83	6889	9·110	0·0120	1·449	4·88
84	7056	9·165	0·0119	1·466	5 25
85	7225	9·220	0·0118	1·484	5·67
86	7396	9·274	0·0116	1·501	6·14
87	7569	9·327	0 0115	1·518	6·69
88	7744	9·381	0·0114	1·536	7·33
89	7921	9·434	0·0112	1·553	8·09
90	8100	9·487	0·0111	1·571	9·00
91	8281	9 539	0 0110	1·588	10·11
92	8464	9·592	0 0109	1·606	11·50
93	8649	9·644	0·0108	1·623	13·29
94	8836	9 695	0·0106	1·641	15·67
95	9025	9·747	0·0105	1·658	19·0
96	9216	9·798	0·0104	1·676	24·0
97	9409	9 849	0·0103	1·693	32·3
98	9604	9·899	0 0102	1·710	49·0
99	9801	9·950	0 0101	1·728	99·0
100	10000	10·000	0·0100	1·745	∞

TABLE 38.

LOGARITHMS TO 4 PLACES.

N	0	1	2	3	4	5	6	7	8	9	Diff.
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	42
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	23
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	[illegible]	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
N.	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE 38—*continued.*

Logarithms to 4 Places.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
58	7634	7642	7649	7657	7764	7672	7679	7686	7694	7701	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	5
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4
N.	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE 38—*continued.*

Logarithms to 4 Places.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
100	00000	0043	0087	0130	0173	0217	0260	0303	0346	0389	43
101	00432	0475	0518	0561	0604	0647	0689	0732	0775	0817	43
102	00860	0903	0945	0988	1030	1072	1115	1157	1199	1242	42
103	01284	1326	1368	1410	1452	1494	1536	1578	1620	1662	42
104	01703	1745	1787	1828	1870	1912	1953	1995	2036	2078	42
105	02119	2160	2202	2243	2284	2325	2366	2407	2449	2490	41
106	02531	2572	2612	2653	2694	2735	2776	2816	2857	2898	40
107	02938	2979	3019	3060	3100	3141	3181	3222	3262	3302	40
108	03342	3383	3423	3463	3503	3543	3583	3623	3663	3703	40
109	03743	3782	3822	3862	3902	3941	3981	4021	4060	4100	40
110	04139	4179	4218	4258	4297	4336	4376	4415	4454	4493	39
N.	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE 39.

Trigonometrical Functions.

	Sine.		Tangent.		Cotangent.		Cosine.		
0°	0·0000	175	0·0000	175	∞		1·0000	02	90
1	·0175	174	·0175	174	57·29		0·9998	04	89
2	·0349	174	·0349	175	28·64		·9994	08	88
3	·0523	175	·0524	175	19·08		·9986	10	87
4	·0698	174	·0699	176	14·30		·9976	14	86
5	·0872	173	·0875	176	11·43		·9962	17	85
6	·1045	174	·1051	177	9·514		·9945	20	84
7	·1219	173	·1228	177	8·144		·9925	22	83
8	·1392	172	·1405	179	7·115	801	·9903	26	82
9	·1564	172	·1584	179	6·314	643	·9877	29	81
10	·1736	172	·1763	181	5·671	526	·9848	32	80
11	·1908	171	·1944	182	5·145	440	·9816	35	79
12	·2079	171	·2126	183	4·705	374	·9781	37	78
13	·2250	169	·2309	184	4·331	320	·9744	41	77
14	·2419	169	·2493	186	4·011	279	·9703	44	76
15	·2588	168	·2679	188	3·732	245	·9659	46	75
16	·2756	168	·2867	190	3·487	216	·9613	50	74
17	·2924	166	·3057	192	3·271	193	·9563	52	73
18	·3090	166	·3249	194	3·078	174	·9511	56	72
19	·3256	164	·3443	197	2·904	157	·9455	58	71
20	·3420	164	·3640	199	2·747	142	·9397	61	70
21	·3584	162	·3839	201	2·605	130	·9336	64	69
22	·3746	161	·4040	205	2·475	119	·9272	67	68
23	·3907	160	·4245	207	2·356	110	·9205	70	67
24	·4067	159	·4452	211	2·246	101	·9135	72	66
25	·4226	158	·4663	214	2·145	95	·9063	75	65
26	·4384	156	·4877	218	2·050	87	·8988	78	64
27	·4540	155	·5095	222	1·963	82	·8910	81	63
28	·4695	153	·5317	226	1·881	77	·8829	83	62
29	·4848	152	·5543	231	1·804	72	·8746	86	61
30	·5000	150	·5774	235	1·732	68	·8660	88	60
31	·5150	149	·6009	240	1·664	64	·8572	92	59
32	·5299	147	·6249	245	1·600	60	·8480	93	58
33	·5446	146	·6494	251	1·540	57	·8387	97	57
34	·5592	144	·6745	257	1·483	55	·8290	98	56
35	·5736	142	·7002	263	1·428	52	·8192	102	55
36	·5878	140	·7265	271	1·376	49	·8090	104	54
37	·6018	139	·7536	277	1·327	47	·7986	106	53
38	·6157	136	·7813	285	1·280	45	·7880	109	52
39	·6293	135	·8098	293	1·235	43	·7771	111	51
40	·6428	133	·8391	302	1·192	42	·7660	113	50
41	·6561	130	·8693	311	1·150	39	·7547	116	49
42	·6691	129	·9004	321	1·111	39	·7431	117	48
43	·6820	127	·9325	332	1·072	36	·7314	121	47
44	·6947	124	·9657	343	1·036	36	·7103	122	46
45	·7071		1·0000		1·000		·7071		45°
	Cosine.		Cotangent.		Tangent.		Sine.		

INDEX

TABLES

THE END

Printed by R. & R. CLARK, *Edinburgh*

ck

Jw